

Smaller wavelength.

• PLANCK'S RADIATION FORMULA

Planck introduced new idea to explain the distribution of energy among the various wavelengths for the cavity radiation. He ~~was~~ also assumed that the atoms of the walls of the cavity radiator ~~becomes~~ behaves as an oscillator and each oscillator with a characteristic frequency of oscillation. These oscillator emits electromagnetic radiant energy in the cavity and also absorb the same to maintain an equilibrium state. Therefore, Planck made two assumptions regarding their atomic oscillation.

(i) An oscillator can have only energies given by

$$E = nh\nu \quad \text{--- (1)}$$

where, ν is the frequency of oscillator and h is the Planck's constant

n is an integer known as "Quantum number"

It means that the oscillator can have only the energies $h\nu$, $2h\nu$, $3h\nu$ and so on.

(ii) The oscillator do not emit or absorb energy continuously but only in jump, that is an oscillator emit and absorbs packet of energy and each packet carrying an amount of energy $h\nu$.

Under the basis of these basic assumptions let us calculate the average energy of Planck's oscillator of frequency ν . According to Boltzmann law the relative probability of an oscillator having the energy

$$e^{-h\nu/kT}$$

where, k = Boltzmann constant

Let us consider $N_0, N_1, N_2, \dots, N_r$ be the number of oscillations having the energy zero, $h\nu$, $2h\nu$, $3h\nu$ respectively. Therefore, the total number of oscillator can be written as

$$N = N_0 + N_1 + N_2 + \dots + N_r$$

$$N = N_0 + N_0 e^{-h\nu/kT} + N_0 e^{-2h\nu/kT} + \dots + N_0 e^{-rh\nu/kT}$$

$$N = N_0 [1 + e^{-h\nu/kT} + e^{-2h\nu/kT} + e^{-3h\nu/kT} + \dots]$$

$$N = \frac{N_0}{1 - e^{-h\nu/kT}} \quad \text{--- (2)}$$

Now the total energy of the oscillator can be written as

$$E = N_0 \times 0 + N_1 h\nu + N_2 2h\nu + N_3 3h\nu + \dots$$

Putting the value of N_1, N_2 and N_3 from the above relations

$$E = h\nu N_0 e^{-h\nu/kT} + 2h\nu \cdot N_0 e^{-2h\nu/kT} + 3h\nu \cdot N_0 e^{-3h\nu/kT} + \dots$$

$$E = N_0 h\nu e^{-h\nu/kT} [1 + 2e^{-h\nu/kT} + 3e^{-2h\nu/kT} + \dots] \quad (3)$$

We know that,

$$1 + 2x + 3x^2 + \dots = \frac{1}{(1-x)^2}$$

Hence the eq (3) becomes

$$E = N_0 h\nu e^{-h\nu/kT} \cdot \frac{1}{(1 - e^{-h\nu/kT})^2} \quad (4)$$

Therefore, the average velocity of an oscillator can be written as

$$\bar{E} = \frac{E}{N}$$

putting the value of E and N from the above equation

$$\bar{E} = \frac{N_0 e^{-h\nu/kT} \cdot \frac{h\nu}{(1 - e^{-h\nu/kT})^2}}{N_0 \cdot \frac{1}{(1 - e^{-h\nu/kT})}}$$

$$\bar{E} = \frac{e^{-h\nu/kT} (1 - e^{-h\nu/kT})}{(1 - e^{-h\nu/kT})^2} h\nu$$

$$\bar{E} = \frac{h\nu}{(1 - e^{-h\nu/kT}) \cdot e^{h\nu/kT}}$$

$$\therefore \bar{E} = \frac{h\nu}{(e^{h\nu/kT} - 1)} \quad (5)$$

According to Rayleigh-Jean's law the density of radiation of frequencies lying between ν and $\nu + d\nu$ is related to the average energy

$$E\nu \cdot d\nu = \frac{8\pi\nu^2}{c^3} (d\nu) \cdot E$$

where, c = velocity of electromagnetic radiation
Putting the value of average energy from the above equation

$$E\nu \cdot d\nu = \frac{8\pi\nu^2}{c^3} \cdot d\nu \cdot \frac{h\nu}{(e^{h\nu/kT} - 1)}$$

$$E\nu \cdot d\nu = \frac{8\pi h\nu^3}{c^3} \cdot \frac{1}{(e^{h\nu/kT} - 1)} \cdot d\nu \quad \text{--- (5)}$$

This is the Planck's radiation formula in terms of frequency

This formula is also can be written in terms of wavelength i.e.

$$c = \nu\lambda$$

$$\therefore \nu = \frac{c}{\lambda}$$

$$\text{and, } d\nu = -\frac{c}{\lambda^2} d\lambda$$

Hence the equation (5) becomes

$$E\lambda \cdot d\lambda = \frac{8\pi h}{c^3} \cdot \frac{c^3}{\lambda^3} \cdot \frac{1}{(e^{hc/\lambda kT} - 1)} \cdot -\frac{c}{\lambda^2} d\lambda$$

$$E\lambda \cdot d\lambda = -\frac{8\pi hc}{\lambda^5} \cdot \frac{d\lambda}{(e^{hc/\lambda kT} - 1)} \quad \text{--- (6)}$$

This is the Planck's radiation formula in terms of wavelength.

- Deduction of Wein's displacement law by Planck's law of radiation :-

According to Planck's law of radiation i.e.

$$E_{\lambda} d\lambda = \frac{-8\pi hc}{\lambda^5} \cdot \frac{d\lambda}{e^{\frac{hc}{\lambda kT}} - 1}$$

$$E_{\lambda} = -8\pi hc \left[\lambda^{-5} \cdot \frac{1}{e^{\frac{hc}{\lambda kT}} - 1} \right] \quad \text{--- (1)}$$

For the maximum value of E_{λ} it is necessary that

$$\frac{dE_{\lambda}}{d\lambda} = 0$$

Putting the value of E_{λ} from eq (1)

$$\frac{d}{d\lambda} \left[-8\pi hc \left\{ \lambda^{-5} \cdot \frac{1}{(e^{\frac{hc}{\lambda kT}} - 1)} \right\} \right] = 0$$

$$\Rightarrow -8\pi hc \frac{d}{d\lambda} \left\{ \lambda^{-5} \cdot \frac{1}{(e^{\frac{hc}{\lambda kT}} - 1)} \right\} = 0$$

$$\Rightarrow -5\lambda^{-6} \cdot (e^{\frac{hc}{\lambda kT}} - 1)^{-1} + \lambda^{-5} (-1) (e^{\frac{hc}{\lambda kT}} - 1)^{-2} \cdot e^{\frac{hc}{\lambda kT}} \left(\frac{hc}{\lambda^2 kT} \right) = 0$$

$$\Rightarrow \frac{-5}{\lambda^6 (e^{\frac{hc}{\lambda kT}} - 1)} + \frac{hc \cdot e^{\frac{hc}{\lambda kT}}}{\lambda^7 kT (e^{\frac{hc}{\lambda kT}} - 1)^2} = 0$$

$$\Rightarrow \frac{hc \cdot e^{\frac{hc}{\lambda kT}}}{\lambda^7 \cdot kT (e^{\frac{hc}{\lambda kT}} - 1)^2} = \frac{5\lambda}{\cancel{\lambda^6} (e^{\frac{hc}{\lambda kT}} - 1)}$$

$$hc \cdot e^{\frac{hc}{\lambda kT}} = 5kT \left(e^{\frac{hc}{\lambda kT}} - 1 \right)$$

$$\frac{hc \cdot e^{\frac{hc}{\lambda kT}}}{\lambda kT (e^{\frac{hc}{\lambda kT}} - 1)} = 5$$

when, $\lambda = \lambda_m$

Hence the above equation becomes:-

$$\frac{hc}{\lambda_m kT} \cdot \frac{e^{\frac{hc}{\lambda_m kT}}}{\left(e^{\frac{hc}{\lambda_m kT}} - 1 \right)} = 5 \quad \text{--- (2)}$$

To solve this expression,
let us consider

$$\frac{hc}{k} = a$$

$$\text{and } \lambda_m \cdot T = b$$

Hence under this consider the eq (2) becomes

$$\frac{a}{b} \cdot \frac{e^{a/b}}{\left(e^{a/b} - 1 \right)} = 5$$

$$\frac{a}{b} = \frac{5(e^{a/b} - 1)}{e^{a/b}}$$

$$\frac{a}{b} = 5(e^{a/b} - 1) \cdot e^{-a/b}$$

$$\frac{a}{b} = 5(1 - e^{-a/b}) \quad \text{--- (3)}$$

But by using the method of approximation the value of $\frac{a}{b}$ can be find out i.e.

$$\frac{a}{b} = 4.9651$$

$$b = \frac{a}{4.9651}$$

Putting the value of a and b

$$\lambda_m \cdot T = \frac{hc}{4.9651 \text{ K}} = \text{constant}$$

$$\boxed{\lambda_m \cdot T = \text{Constant}}$$

which is Wien's displacement law hence Wien's displacement law is deduced by Planck's law of radiation.

• Deduction of Stefan's law by Planck's law of radiation:-

According to Planck's law of radiation the energy density inside the black body enclosure between the frequency ν and $\nu + d\nu$

$$\boxed{U_{\nu} d\nu = \frac{8\pi h}{c^3} \frac{\nu^3 d\nu}{(e^{\frac{h\nu}{kT}} - 1)}} \quad \text{--- (1)}$$

Therefore the total energy density inside the enclosure can be obtained by integrating from 0 to ∞ (Infinity) i.e.

$$U = \int_0^{\infty} U_{\nu} d\nu$$

$$U = \int_0^{\infty} \frac{8\pi h}{c^3} \frac{\nu^3 d\nu}{(e^{\frac{h\nu}{kT}} - 1)}$$

$$U = \frac{8\pi h}{c^3} \int_0^{\infty} \frac{v^3}{\left(e^{\frac{h\nu}{kT}} - 1\right)} \cdot d\nu$$

(2)

$$\frac{h\nu}{kT} = x$$

$$\nu = \frac{xkT}{h} = \frac{kT}{h} \cdot x$$

$$d\nu = \frac{kT}{h} dx$$

Putting all this value in eq (2)

$$U = \frac{8\pi h}{c^3} \int_0^{\infty} \frac{\left(\frac{kT}{h} x\right)^3}{e^x - 1} \cdot \frac{kT}{h} \cdot dx$$

$$= \frac{8\pi h}{c^3} \left(\frac{kT}{h}\right)^4 \int_0^{\infty} \frac{x^3}{e^x - 1} \cdot dx \quad (3)$$

But the value of standard integration

$$\int_0^{\infty} \frac{x^3}{e^x - 1} \cdot dx = \frac{\pi^4}{15}$$

Hence the eq (3) becomes

$$U = \frac{8\pi h}{c^3} \left(\frac{kT}{h}\right)^4 \cdot \frac{\pi^4}{15}$$

Here

$$U = \left(\frac{8\pi^5 k^4}{15c^3 h^3}\right) \cdot T^4$$

In this expression $\frac{8\pi^5 k^4}{15c^3 h^3} = \sigma$
known as Stefan's constant

which is stefan's law
Hence stefan's law is deduced from planck's
law of radiation.

Questions