Thm For any Boolean Algebra B prove that

(i) $a \le b$, $a \le c \Rightarrow a \le bc$

(ii) $a \le b \Rightarrow a \le b + c$

(iii) $a \le b \Leftrightarrow b' \le a'$

(iv) $0 \le a \le 1 \quad \forall a \in B$

Proof

(i) Let $a \le b$ & $a \le c$

$\Rightarrow a \cdot b' = 0 \quad a \cdot c' = 0$

$\Rightarrow a \cdot b' + a \cdot c' = 0$

$\Rightarrow a \cdot (b' + c') = 0$

$\Rightarrow a \cdot (b \cdot c)' = 0$

$\Rightarrow a \le bc$ proved

(ii) Let $a \le b$

$\Rightarrow a \cdot b' = 0 \quad — (1)$

Now $a \cdot (b+c)' = a \cdot (b' \cdot c')$

$= (a \cdot b') \cdot c'$

$= 0 \cdot c'$

$= 0$

$\Rightarrow a \le b + c$ proved

(iii) Let $a \le b$

$\Leftrightarrow ab' = 0$

$\Leftrightarrow b' \cdot a = 0$

$\Leftrightarrow b' \cdot (a')' = 0$

$\Leftrightarrow b' \le a'$ proved

Teacher's Signature........................

Date
Page No.

(iv) For every $a \in B$ we have

$$0 \cdot a' = 0$$

$$\Rightarrow 0 \leq a \quad \text{--- (1)}$$

& $$a \cdot 1' = a \cdot 0$$
$$= 0$$

$$\Rightarrow a \leq 1 \quad \text{--- (2)}$$

From (1) & (2) $0 \leq a \leq 1 \ \forall a \in B$.

EXAMPLES

Ex (1) In a Boolean algebra B prove that
$$a' + b = 1 \iff a + b = b \quad \forall a, b \in B.$$

Proof Let $a + b = b$ --- (1)

Now $a' + b = a' + (a+b)$ From (1)
$$= (a' + a) + b$$
$$= 1 + b$$
$$= 1$$

Conversely let $a' + b = 1$

Now $a + b = (a+b) \cdot 1$
$$= (a+b) \cdot (a' + b)$$
$$= (b + a) \cdot (b + a')$$
$$= b + (a \cdot a')$$
$$= b + 0$$
$$= b$$

Proved

Teacher's Signature..................

Ex② For any Boolean algebra B prove that
$$(a+b)\cdot(b+c)\cdot(c+a) = a\cdot b + b\cdot c + c\cdot a$$

solⁿ

$$\begin{aligned}
LHS &= (a+b)\cdot(b+c)\cdot(c+a) \\
&= (b+a)\cdot(b+c)\cdot(c+a) \\
&= \{b+(a\cdot c)\}\cdot(c+a) \\
&= (b+a\cdot c)\cdot c + (b+a\cdot c)\cdot a \\
&= c\cdot(b+a\cdot c) + a\cdot(b+a\cdot c) \\
&= c\cdot b + c\cdot(ac) + a\cdot b + a\cdot(a\cdot c) \\
&= b\cdot c + c\cdot(c\cdot a) + a\cdot b + a\cdot(a\cdot c) \\
&= b\cdot c + (c\cdot c)\cdot a + a\cdot b + (a\cdot a)\cdot c \\
&= b\cdot c + c\cdot a + a\cdot b + a\cdot c \\
&= b\cdot c + c\cdot a + a\cdot b + c\cdot a \\
&= a\cdot b + b\cdot c + c\cdot a + c\cdot a \\
&= a\cdot b + b\cdot c + c\cdot a \\
&= RHS
\end{aligned}$$
proved

Ex③ In a Boolean algebra B prove that
$$[x'\cdot(x+y)]' + [y\cdot(y+x')]' + [y\cdot(y'+x)]' = 1$$

solⁿ

$$\begin{aligned}
LHS &= [x'\cdot(x+y)]' + [y\cdot(y+x')]' + [y\cdot(y'+x)]' \\
&= (x')' + (x+y)' + y' + (y+x')' + y' + (y'+x)' \\
&= x + x'\cdot y' + y' + y'\cdot x + y' + y\cdot x' \\
&= x + (y'+y') + y'\cdot x' + y'\cdot x + x'\cdot y \\
&= x + y' + y'\cdot(x'+x) + x'\cdot y \\
&= x + y' + y'\cdot 1 + y\cdot x' \\
&= x + y' + y' + y\cdot x'
\end{aligned}$$

Teacher's Signature.....................

Date			
Page No.			

$= x + y' + y \cdot x' = x + x' \cdot y + y'$

$= (x + x') \cdot (x + y) + y' = 1(x+y) + y'$

$\Rightarrow x + y + y' = x + 1$

$= 1 = RHS$ Proved

Ex (4) In Boolean algebra B prove that

$$a + b \leq c \Leftrightarrow a \leq c \ \& \ b \leq c$$

Sol^n Let $a \leq c$ & $b \leq c$

$\Rightarrow a + c = c$ & $a \cdot c = a$ and $b + c = c$ & $b \cdot c = b$

Now $(a+b) + c = a + (b + c) = a + c = c$

ie $(a+b) + c = c$

$\Rightarrow a + b \leq c$

Conversely let $a + b \leq c$

$\Rightarrow (a+b) \cdot c = a + b$

$\Rightarrow a \cdot c + b \cdot c = a + b$

$\Rightarrow a \cdot c = a$ & $b \cdot c = b$

$\Rightarrow a \leq c$ & $b \leq c$

Proved