Tensor Calculus

Some important definition:

1. Summation Convention: The expression $a_1 x^1 + a_2 x^2 + \dots + a_n x^n$ is represented by $\sum_{i=1}^{n} a_i x^i$. Summation convention means drop the sigma sign and adopt the convention:

$$\sum_{i=1}^{n} a_i x^i = a^i x^i$$

2. Dummy suffix: If a suffix occurs twice in a term, once in the upper position and once in the lower position, then that suffix is called dummy suffix. e.g. i is the dummy suffix in $a_i^{\mu} x^i$.
3. Real suffix: A suffix which is not repeated is called a real suffix. In $a_i^{\mu} x^i$, μ is a real suffix. A real suffix can not be replaced by another real suffix. For example: $a_i^{\mu} x^i \neq a_i^{\nu} x^i$
4. Kronecker delta: It is denoted by δ_{ij} and is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Properties: (i) $\frac{\partial x^i}{\partial x^j} = \delta^i_j$, (ii) $\delta^i_j A^j = A^i$,

(iii) $\delta^i_i = 4$, for $\delta^i_i = \delta^1_1 + \delta^2_2 + \delta^3_3 + \delta^4_4 = 1+1+1+1 = 4$

(iv) $\delta^i_j \delta^j_k = \delta^i_k$.

5. World vectors: We extend ordinary vector analysis (3-vector) to four dimensions (i.e 4 vectors). These four dimensional vectors are called four vectors or world vectors. Since four dimensional coordinates are orthogonal and so, $i \cdot i = j \cdot j = k \cdot k = p \cdot p = 1$ and $i \cdot j = j \cdot k = k \cdot p = p \cdot i = p \cdot j = k \cdot j = 0$

Sd/- 13/5/2020

The world vectors are denoted by bars beneath their symbols. If A and B are two world vectors, then

$$A = A_1 i + A_2 J + A_3 k + A_4 p$$

and $$B = B_1 i + B_2 J + B_3 k + B_4 p$$

The scalar product A and B is defined as

$$\underline{A} \cdot \underline{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 - A_4 B_4$$

$$\Rightarrow \underline{A} \cdot \underline{B} = \bar{A} \cdot \bar{B} - A_4 B_4$$, where A and B are ordinary vectors.

Thus $\underline{A} = \underline{A}(A_1, A_2, A_3, iA_4)$, $i = \sqrt{-1}$

Hence $\underline{A}^2 = \underline{A} \cdot \underline{A} = A_1^2 + A_2^2 + A_3^2 - A_4^2$

or $A^2 = \bar{A}^2 - A_4^2$, where $\bar{A} = \bar{A}(A_1, A_2, A_3)$ is ordinary three dimensional vectors.

The world vector A is said to be space like if $A^2 > 0$ and time like if $A^2 < 0$.

6. ∇ (del operator) $= i\dfrac{\partial}{\partial x} + J\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z}$

$\Box$ (D'Alembertian operator), It is a four dimensional operator, defined as

$$\Box = i\frac{\partial}{\partial x} + J\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} + p\frac{\partial}{\partial p}$$

$$\text{Div } A = \Box \cdot A = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} + \frac{\partial A_4}{\partial p}$$

$$\text{Curl } A = \Box \times A = \begin{vmatrix} i & J & k & p \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial p} \\ A_1 & A_2 & A_3 & A_4 \end{vmatrix}$$

S.A. Hashmi
KCC
13.05.20