Partial Differential Equation

An equation containing partial derivatives of one or more variables with respect to their independent variables is called partial differential eqn.

Example $z = f(x, y)$ is a function in which x and y are independent variables & z is dependent variable.

We will use the following notation

$$p = \frac{\partial z}{\partial x} \quad q = \frac{\partial z}{\partial y} \quad r = \frac{\partial^2 z}{\partial x^2} \quad s = \frac{\partial^2 z}{\partial x \partial y} \; \& \; t = \frac{\partial^2 z}{\partial y^2}$$

Formation of partial differential equation

① By eliminating arbitrary constants.

Let given equation be $f(x, y, z, a, b) = 0$ —— ①

Differentiating ① partially with respect to x we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

ie $$\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} = 0 \quad \text{—— ②}$$

Differentiating ① partially with respect to y we get

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$$

ie $$\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z} = 0 \quad \text{—— ③}$$

There are two constants a & b which can be eliminating using eqn ① ② & ③. On eliminating a & b we get an eqn in $F(x, y, z, p \,\&\, q) = 0$ —— ④

Which required partial differential eqn.

Example: Form a partial differential eqn be eliminating h & k from $(x-h)^2+(y-k)^2+z^2=a^2$.

Sol^n Given $(x-h)^2+(y-k)^2+z^2=a^2$ —— (1)

Differentiating (1) partially w.r.to x we get

$$2(x-h)+0+2z\frac{\partial z}{\partial x}=0$$

ie $(x-h)+zp=0$ —— (2)

Differentiating (1) partially w.r.to y we get

$$0+2(y-k)^2+2z\frac{\partial z}{\partial y}=0$$

ie $(y-k)^2+zq=0$ —— (3)

Putting the value of $x-h$ & $y-k$ from (2) & (3) in eqn (1) we get

$$(-zp)^2+(-zq)^2+z^2=a^2$$

$$\Rightarrow z^2p^2+z^2q^2+z^2=a^2$$

$$\Rightarrow z^2(p^2+q^2+1)=a^2$$

(Required diff. eqn.)

Example Form a differential equation by eliminating a & b from $z=(x+a)(y+b)$

Sol^n Given $z=(x+a)(y+b)$ —— (1)

Diff. (1) partially w.r.to x we get

$$\frac{\partial z}{\partial x}=(y+b)$$

ie $p=(y+b)$ —— (2)

Differentiating (1) partially w.r.to y we get

$$\frac{\partial z}{\partial y} = (x+a)$$

ie $q = (x+a)$ —— (3)

Putting the value of $(x+a)$ & $(y+b)$ from (3) & (2) in eqn (1) we get

$$z = pq \quad \text{(Required diff. eqn)}$$

(2) <u>By eliminating arbitrary function.</u>

Let u & v are two functions of x, y & z which are connected by the relation

$$f(u,v) = 0 \quad \text{—— (1)}$$

Diff. (1) partially w.r.to x we get

$$\frac{\partial f}{\partial u}\cdot\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

ie $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \quad \text{—— (2)}$$

Similarly differentiating (1) partially w.r.to y we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0 \quad \text{—— (3)}$$

Eliminating $\frac{\partial f}{\partial u}$ & $\frac{\partial f}{\partial v}$ from (2) & (3) we get

$$\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) - \left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

$$\Rightarrow \left(\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} - \frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial z}\right)p + \left(\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial z} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}\right)q$$

$$= \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\right)$$

which is of the form

$$Pp + Qq = R \quad \text{——— (4)}$$

where $P = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial z} = \frac{\partial(u,v)}{\partial(y,z)}$

$$Q = \frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial z} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z} = \frac{\partial(u,v)}{\partial(z,x)}$$

$$R = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x} = \frac{\partial(u,v)}{\partial(x,y)}$$

Example Form a partial differential eqn by eliminating arbitrary function f from $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$.

Soln Given $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ ——— (1)

Differentiating (1) partially w.r.to x we get

$$\frac{\partial z}{\partial x} = 0 + 2f'\left(\frac{1}{x} + \log y\right)\cdot\left(-\frac{1}{x^2}\right)$$

$$\Rightarrow p = -\frac{2}{x^2} f'\left(\frac{1}{x} + \log y\right)$$

$$\text{ie } -px^2 = 2f'\left(\frac{1}{x} + \log y\right) \quad \text{——— (2)}$$

Diff. eqn (1) partially w.r.to y we get

$$\frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)\cdot\frac{1}{y}$$

$$\Rightarrow q = 2y + \frac{2}{y}f'\left(\frac{1}{x} + \log y\right)$$

$$\Rightarrow qy = 2y^2 + 2f'\left(\frac{1}{x} + \log y\right)$$

$$\Rightarrow qy - 2y^2 = 2f'\left(\frac{1}{x} + \log y\right) \quad \text{——— (3)}$$

From (2) & (3)

$$qy - 2y^2 = -px^2$$

$$\Rightarrow px^2 + qy = 2y^2 \quad \text{(Reqd. diff. eqn).}$$