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for (2-4)

CC-614

Laplace Transform

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$f(p) = \int_0^\infty e^{-pt} F(t)\,dt$ is called the Laplace transform of the function $F(t)$, denoted by $L\{F(t)\}$, p is parameter.

NOTE: (i) $L\{F(t)\}$ = a function of new variable p

$$= f(p)$$
$$= \int_0^\infty e^{-pt} F(t)\,dt.$$

$L\{F(t)\}$ is said to be exists iff the integral $f(p) = \int_0^\infty e^{-pt} F(t)\,dt$ is convergent.

(ii) $f(s) = \int_0^\infty e^{-st} F(t)\,dt$

$$= L\{F(t)\}$$

NOTE: (i) $L\{F(t)\}$ is linear

$\therefore L\{a_1 F_1(t) + a_2 F_2(t)\} = a_1 L\{F_1(t)\} + a_2 L\{F_2(t)\}$

Table of Formula

	$F(t)$	$L\{F(t)\}$
1	1	$1/p$, $p>0$
2	t^n (n is a +ve integer)	$\frac{\lfloor n}{p^{n+1}}$, $p>0$
3.	t^a $(a>-1)$	$\Gamma(a+1)/p^{a+1}$, $p>0$
4.	e^{at}	$\frac{1}{p-a}$, $p>a$

5.	$\sin at$	$\frac{a}{p^2+a^2}$, $p>0$
6.	$\cos at$	$\frac{p}{p^2+a^2}$, $p>0$
7.	$\sinh at$	$\frac{a}{p^2-a^2}$, $p>\|a\|$
8.	$\cosh at$	$\frac{p}{p^2-a^2}$, $p>\|a\|$

Proof of formula

1) Find the Laplace transform of $F(t)=1$.
ie; $L\{1\}=?$

Ans: We have

$$L\{F(t)\}=\int_0^\infty e^{-pt}F(t)\,dt$$

$$\therefore L\{1\}=\int_0^\infty e^{-pt}\cdot 1\,dt$$

$$=\int_0^\infty e^{-pt}\,dt$$

$$=\left[\frac{-e^{-pt}}{p}\right]_0^\infty=-\frac{1}{p}\left[0-1\right]=\frac{1}{p}\quad \text{if } p>0$$

2) Find $L\{t^n\}$, n being +ve integer

Ans: We have,

$$L\{F(t)\}=\int_0^\infty e^{-pt}F(t)\,dt$$

$$\therefore L\{t^n\}=\int_0^\infty e^{-pt}\cdot t^n\,dt$$

$$=\left[t^n\frac{e^{-pt}}{-p}\right]_0^\infty-\int_0^\infty nt^{n-1}\frac{e^{-pt}}{-p}\,dt \quad \text{[Int. by parts}$$

$$=-\frac{1}{p}\left[\frac{t^n}{e^{pt}}\right]_0^\infty+\frac{n}{p}\int_0^\infty t^{n-1}e^{-pt}\,dt \quad \text{[By L.H.}$$

$\therefore L\{t^n\} = -\frac{1}{p}\lim_{t\to\infty}\frac{t^n}{e^{pt}} + 0 - \frac{n}{p}\int_0^\infty t^{n-1}e^{-pt}dt$

$= -0 + 0 + \frac{n}{p} L\{t^{n-1}\}$ [By L-H Rule]

$= \frac{n}{p}\frac{n-1}{p} L\{t^{n-2}\}$

Proceeding in this way we get

$\therefore L\{t^n\} = \frac{n}{p}\cdot\frac{n-1}{p}\cdot\frac{n-2}{p}\cdot \cdots \frac{n-(n-1)}{p} L\{t^0\}$

$= \frac{n}{p}\cdot\frac{n-1}{p}\cdot\frac{n-2}{p}\cdot \cdots \frac{1}{p}\cdot L\{1\}$

$= \frac{\lfloor n}{p^n}\frac{1}{p}$ if $p>0$ $\{\because L\{1\} = \frac{1}{p}$ if $p>0\}$

$= \frac{\lfloor n}{p^{n+1}}$

or $L\{t^n\} = \frac{\Gamma(n+1)}{p^{n+1}}$ if $p>0$.

③ Find $L\{e^{at}\}$

Ans:- $\therefore L\{e^{at}\} = \int_0^\infty e^{-pt}\cdot e^{at}dt$

$= \int_0^\infty e^{(-p+a)t}dt$

$= \int_0^\infty e^{-(p-a)t}dt = \left[\frac{e^{-(p-a)t}}{-(p-a)}\right]_0^\infty$

$= \frac{-1}{(p-a)}[0-1]$ if $p>a$

$\therefore L\{e^{at}\} = \frac{1}{p-a}$ if $p>a$.

(4) Find (i) $L\{\sin at\}$ & (ii) $L\{\cos at\}$

Ans.

(i) $L\{\sin at\} = \int_0^\infty e^{-pt}.\sin at \, dt$ ——— (1)

$= \left[\sin at \frac{e^{-pt}}{-p}\right]_0^\infty - \int_0^\infty a\cos at \frac{e^{-pt}}{-p} dt$ [Int. by parts]

$= -\frac{1}{p}[0-0] + \frac{a}{p}\int_0^\infty \cos at \, e^{-pt} dt$ if $p>0$

$= \frac{a}{p}\int_0^\infty e^{-pt}\cos at \, dt$

$= \frac{a}{p}\left[\cos at \frac{e^{-pt}}{-p}\right]_0^\infty + \frac{a}{p}\int_0^\infty a\sin at \frac{e^{-pt}}{-p} dt$

$= -\frac{a}{p^2}[0-1] - \frac{a^2}{p^2}\int_0^\infty e^{-pt}\sin at \, dt$ if $p>0$

$= \frac{a}{p^2} - \frac{a^2}{p^2} L\{\sin at\}$ {Using (1)}

$\Rightarrow L \sin at \left\{1+\frac{a^2}{p^2}\right\} = \frac{a}{p^2} \Rightarrow L \sin at \left\{\frac{p^2+a^2}{\cancel{p^2}}\right\} = \frac{a}{\cancel{p^2}}$

$\Rightarrow L\{\sin at\} = \frac{a}{p^2+a^2}$ if $p>0$

(ii) $L\{\cos at\} = \int_0^\infty e^{-pt}.\cos at \, dt$

$= \left[\cos at . \frac{e^{-pt}}{-p}\right]_0^\infty + \int_0^\infty a\sin at \frac{e^{-pt}}{-p} dt$ [Int by parts]

$= -\frac{1}{p}[0-1] - \frac{a}{p}\int_0^\infty e^{-pt}\sin at \, dt$ if $p>0$

$= \frac{1}{p} - \frac{a}{p}\int_0^\infty e^{-pt}\sin at \, dt$

$= \frac{1}{p} - \frac{a}{p}\left[\sin at \frac{e^{-pt}}{-p}\right]_0^\infty + \frac{a}{p}\int_0^\infty a\cos at \frac{e^{-pt}}{-p} dt$

$\therefore L\{\cos at\} = \frac{1}{p} - \frac{a}{p}[0-0] - \frac{a^2}{p^2}\int_0^\infty e^{-pt}\cos at\, dt$ if $p>0$

$= \frac{1}{p} - \frac{a^2}{p^2} L\{\cos at\}$

$\Rightarrow L\{\cos at\}\left(1+\frac{a^2}{p^2}\right) = \frac{1}{p}$

$\Rightarrow L\{\cos at\}\left\{\frac{p^2+a^2}{p^{\cancel{2}}}\right\} = \frac{1}{\cancel{p}}$

$\Rightarrow L\{\cos at\} = \frac{p}{p^2+a^2}$ if $p>0$

$\longrightarrow$

Problems: Type-I

(1) $L\{\sin^2 at\}$

Ans:

$L\{\sin^2 at\} = L\left\{\frac{2\sin^2 at}{2}\right\} = L\left\{\frac{1-\cos 2at}{2}\right\}$

$= \frac{1}{2}\left[L\{1\} - L\{\cos 2at\}\right]$

$= \frac{1}{2}\left[\frac{1}{p} - \frac{p}{p^2+4a^2}\right] = \frac{1}{2}\left[\frac{\cancel{p^2}+4a^2-\cancel{p^2}}{p(p^2+4a^2)}\right]$

$= \frac{1}{2}\frac{4a^2}{p(p^2+4a^2)}$

$= \frac{2a^2}{p(p^2+4a^2)}$ Ans

(2) Find (i) $L\{\cosh at\}$

(ii) $L\{\sinh at\}$

Ans:

(i) $L\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\}$

$= \frac{1}{2}\left[L\{e^{at}\} + L\{e^{-at}\}\right]$

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$$\therefore L\{\cosh at\} = \frac{1}{2}\left[\frac{1}{p-a} + \frac{1}{p+a}\right] \quad \text{if } p>|a|$$

$$= \frac{1}{2}\left[\frac{p+\cancel{a}+p-\cancel{a}}{(p-a)(p+a)}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}p}{(p-a)(p+a)}\right]$$

$$= \frac{p}{(p-a)(p+a)} = \frac{p}{p^2-a^2} \quad \text{if } p>|a|$$

(ii) $$L\{\sinh at\} = L\left\{\frac{e^{at}-e^{-at}}{2}\right\}$$

$$= \frac{1}{2}L\{e^{at}\} - \frac{1}{2}\{e^{-at}\}$$

$$= \frac{1}{2}\left[\frac{1}{p-a} - \frac{1}{p+a}\right] \quad \text{if } p>|a|$$

$$= \frac{1}{2}\left[\frac{\cancel{p}+a-\cancel{p}+a}{(p-a)(p+a)}\right]$$

$$= \frac{1}{\cancel{2}}\frac{\cancel{2}a}{p^2-a^2} = \frac{a}{p^2-a^2} \quad \text{if } p>|a|$$

(3) Find $L\{\sin t \cos t\}$

Ans:-

$$\therefore L\{\sin t\cos t\} = L\left\{\frac{1}{2}2\sin t\cos t\right\}$$

$$= \frac{1}{2}L\{\sin 2t\}$$

$$= \frac{1}{\cancel{2}}\frac{\cancel{2}}{p^2+4} \quad \text{if } p>2$$

$$= \frac{1}{p^2+4} \quad \text{if } p>2$$