Problems to solved

① Form differential equation by eliminating a, b, c from $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

② Form a partial differential equation by eliminating function f and F from $z=f(x+iy)+F(x-iy)$

③ Form differential equation by eliminating the arbitrary constants.

(a) $z=ax+a^2y^2+b$ (b) $z=ax+by+ab$

(c) $az+b=a^2x+y$ (d) $z=a(x+y)+b$

④ Form differential equation by eliminating arbitrary functions

(a) $z=F(x^2+y^2)$ (b) $z=e^{my}\phi(x-y)$

(c) $z=f(x)+x\,g(y)$ (d) $f(x+y+z,\ x^2+y^2-z^2)=0$.

Some Definitions

① Complete Integral: If from the partial differential equation $f(x,y,z,p,q)=0$, we can find a relation $F(x,y,z,a,b)=0$ which contains as many arbitrary constant as there are independent variables, the relation $F(x,y,z,a,b)=0$ is called complete integral of the given differential eqn.

② Particular Integral: If some particular values are assigned to the arbitrary constants occurring in complete integral, we get particular solution.

③ Singular Integral: The equation of envelop of the surfaces represented by the complete integral, of given partial differential equation is called singular integral.

④

④ General Integral: If two functions u & v of x, y, z are connected by arbitrary function $f(u,v)=0$ then eliminating f we get partial differential equation of the form $Pp + Qq = R$ —— ①

The solution of eqn ① is $f(u,v)=0$ which is called general integral of partial differential eqn.

Lagrange's Linear Equation

A partial differential equation of the form $Pp+Qq=R$ where P, Q & R are functions of x, y & z is called Lagrange's Linear Equation.

Lagrange's solution of the Linear Equation

We know that, by eliminating f from $f(u,v)=0$ —— ① we get the partial differential eqn $Pp+Qq=R$ ←—— ②

where $P = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}$

$$Q = \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}$$

$$R = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

Thus $f(u,v)=0$ is general integral of $Pp+Qq=R$.

So find general integral we need to find u & v.

Let $u=a$ and $v=b$ be two equations where a & b are arbitrary constants.

Differentiating these two eqns we get

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}\cdot dz = 0 \quad \text{—— ③}$$

$$\& \quad \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0 \quad \text{—— ④}$$

Solving eqn 3 & 4 we get

$$\frac{dx}{\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}}$$

ie we get $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ —— 5

On solving above simultaneous differential equations we get $u=a$ & $v=b$.

Thus solution of eqn 2 is found as $f(u,v)=0$

The eqns 5 are called Lagrange's Auxiliary Equations. or subsidiary eqn

So Working Method is:

To find the solution of partial differential equation $Pp+Qq=R$ —— 1

we form auxiliary equation $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ —— 2

We find two independent integrals of auxiliary equations 2 in the form $u=a$ & $v=b$.

Then the general integral is given by

$$f(u,v)=0$$

Where f is arbitrary function.

Geometrical Interpretation of Lagrange's Linear Equations:

The equation $Pp+Qq=R$ and $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ define the same set of surfaces, and are thus equivalent.