④ Find $L\{\cosh^2 2t\}$

Ans:-

$$L\{\cosh^2 2t\} = L\{\tfrac{1}{2}\, 2\cosh^2 2t\}$$

$$= \frac{1}{2} L\{1+\cosh 4t\}$$

$$= \frac{1}{2} L\{1\} + \frac{1}{2} L\{\cosh 4t\}$$

$$= \frac{1}{2}\frac{1}{p} + \frac{1}{2}\frac{p}{p^2-16} \quad \text{if } p>4$$

$$= \frac{1}{2}\left[\frac{p^2-16+p^2}{p(p^2-16)}\right]$$

$$= \frac{1}{2}\left[\frac{2p^2-16}{p(p^2-16)}\right]$$

$$= \frac{p^2-8}{p(p^2-16)} \quad \text{if } p>4.$$

⑤ Find $L\{7e^{2t}+9e^{-2t}+5\cos t+7t^3+5\sin 3t+2\}$

Ans.

$$L\{7e^{2t}+9e^{-2t}+5\cos t+7t^3+5\sin 3t+2\}$$

$$= 7L\{e^{2t}\}+9L\{e^{-2t}\}+5L\{\cos t\}+7L\{t^3\}+5L\{\sin 3t\} + 2L\{1\}$$

$$= 7\frac{1}{p-2}+9\frac{1}{p+2}+5\frac{p}{p^2+1}+7\frac{\lfloor 3}{p^{3+1}}+5\frac{3}{p^2+9}+\frac{2}{p}$$

$$= \frac{7p+14+9p-18}{p^2-4}+\frac{5p}{p^2+1}+\frac{7\times 6}{p^4}+\frac{15}{p^2+9}+\frac{2}{p}$$

$$= \frac{16p-4}{p^2-4}+\frac{5p}{p^2+1}+\frac{42}{p^4}+\frac{15}{p^2+9}+\frac{2}{p} \quad \text{Ans}$$

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⑥ Find the Laplace transform of the following function

$$3t^4-2t^3+4e^{-3t}-2\sin 5t+3\cos 3t$$

Ans:- Let $F(t) = 3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 3t$

$\therefore L\{F(t)\} = L\{3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 3t\}$

$= 3L\{t^4\} - 2L\{t^3\} + 4L\{e^{-3t}\} - 2L\{\sin 5t\} + 3L\{\cos 3t\}$

$= 3\frac{\lfloor 4}{p^{4+1}} - 2\frac{\lfloor 3}{p^{3+1}} + 4\frac{1}{p+3} - 2\frac{5}{p^2+25} + 3\frac{p}{p^2+9}$

$= \frac{72}{p^5} - \frac{12}{p^4} + \frac{4}{p+3} - \frac{10}{p^2+25} + \frac{3p}{p^2+9}$ Ans

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(7) $2e^{3t} - e^{-3t}$

Ans: Let $F(t) = 2e^{3t} - e^{-3t}$

$\therefore L\{F(t)\} = L\{2e^{3t} - e^{-3t}\}$

$= 2L\{e^{3t}\} - L\{e^{-3t}\}$

$= 2\frac{1}{p-3} - \frac{1}{p+3} = \frac{2p+6-p+3}{p^2-9}$

$= \frac{p+9}{p^2-9}$ Ans

(8) Evaluate function $L\{e^{-2t} - e^{-3t}\}$. Define the validity of the function

Ans:-

$L\{e^{-2t} - e^{-3t}\} = L\{e^{-2t}\} - L\{e^{-3t}\}$

$= \frac{1}{p+2} - \frac{1}{p+3} = \frac{p+3-p-2}{(p+2)(p+3)}$

$= \frac{1}{(p+2)(p+3)}$

$\therefore L\{e^{-2t} - e^{-3t}\}$ is not valid when $p=-2$ or $p=-3$ and valid for all other value of p.

Problems Type-II

(1) Find $L\{F(t)\}$ where

$$F(t) = \begin{bmatrix} 0 & , & 0<t<1 \\ t & , & 1<t<2 \\ 0 & , & t>2 \end{bmatrix}$$

Ans: Here $F(t)$ is not defined at $t=0, t=1$ & $t=2$

$$\therefore L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$$

$$= \int_0^1 e^{-pt}\cdot 0\,dt + \int_1^2 e^{-pt}\cdot t\,dt + \int_2^\infty e^{-pt}\cdot 0\,dt$$

$$= \int_1^2 e^{-pt}\cdot t\,dt$$

$$= \left[t\frac{e^{-pt}}{-p}\right]_1^2 - \int_1^2 1\cdot\frac{e^{-pt}}{-p}\,dt \quad \text{[Int. by parts]}$$

$$= -\frac{1}{p}\left[2e^{-2p} - e^{-p}\right] + \frac{1}{p}\int_1^2 e^{-pt}\,dt$$

$$= -\frac{2e^{-2p}}{p} + \frac{e^{-p}}{p} + \frac{1}{p}\left[\frac{e^{-pt}}{-p}\right]_1^2$$

$$= -2\frac{e^{-2p}}{p} + \frac{e^{-p}}{p} - \frac{1}{p}\left[e^{-2p} - e^{-p}\right]$$

$$= -2\frac{e^{-2p}}{p} + \frac{e^{-p}}{p} - \frac{e^{-2p}}{p^2} + \frac{e^{-p}}{p^2}$$

$$= \left(\frac{1}{p} + \frac{1}{p^2}\right)e^{-p} - \left(\frac{2}{p} + \frac{1}{p^2}\right)e^{-2p} \quad \text{if } p>0$$

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(2) Find the Laplace transform of the function $F(t)$ where,

$$F(t) = \begin{Bmatrix} 4 & , 0<t<1 \\ 3 & , t>1 \end{Bmatrix}.$$

Ans:- Here $F(t)$ is not defined on $t=0, t=1$

$$\therefore L\{F(t)\} = \int_0^\infty e^{-pt} \cdot F(t)\,dt$$

$$= \int_0^1 e^{-pt} \cdot 4\,dt + \int_1^\infty e^{-pt} \cdot 3\,dt$$

$$= 4\left[\frac{e^{-pt}}{-p}\right]_0^1 + 3\left[\frac{e^{-pt}}{-p}\right]_1^\infty$$

$$= \frac{4}{-p}\left[e^{-p}-1\right] - \frac{3}{p}\left[0-e^{-p}\right] \quad \text{if } p>0$$

$$= -\frac{4}{p}e^{-p} + \frac{4}{p} + \frac{3}{p}e^{-p}$$

$$= \frac{4}{p} - \frac{e^{-p}}{p} = \frac{1}{p}(4-e^{-p}), \quad p>0$$

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(3) Find the laplace transform of the function $f(t)$ where,

$$F(t) = \begin{cases} 2t & , 0 \le t < 5 \\ 1 & , t > 5 \end{cases}$$

Ans.

Here $F(t)$ is not defined on $t=5$

$$\therefore L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$$

$$= \int_0^5 e^{-pt} \cdot 2t\,dt + \int_5^\infty e^{-pt} \cdot 1\,dt$$

$$= 2\int_0^5 e^{-pt} \cdot t\,dt + \int_5^\infty e^{-pt}\,dt$$

$$= 2\left[t\,\frac{e^{-pt}}{-p}\right]_0^5 - 2\int_0^5 1 \cdot \frac{e^{-pt}}{-p}\,dt + \left[\frac{e^{-pt}}{-p}\right]_5^\infty$$

$$\therefore L\{F(t)\} = 2\left[5\frac{e^{-5p}}{-p} - 0\right] + \frac{2}{p}\left[\frac{e^{-pt}}{-p}\right]_0^5 + \frac{1}{p}\left[0 - e^{-5p}\right]$$

$$= \frac{10e^{-5p}}{-p} - \frac{2}{p^2}\left[e^{-5p} - 1\right] + \frac{e^{-5p}}{p}$$

$$= -\frac{10e^{-5p}}{p} - \frac{2}{p^2}e^{-5p} + \frac{2}{p^2} + \frac{e^{-5p}}{p}$$

$$= \frac{2}{p^2}(1 - e^{-5p}) - \frac{9}{p}e^{-5p} \qquad p > 0.$$

Ans.

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(4) Find the laplace transform of the function $F(t)$ where

$$F(t) = \begin{cases} \sin t & ; \quad 0 < t < \pi \\ 0 & ; \quad t > \pi \end{cases}$$

Ans:-

Here $F(t)$ is not defined on $t = 0$ and $t = \pi$

$$\therefore L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\, dt$$

$$= \int_0^\pi e^{-pt} \sin t\, dt + \int_\pi^\infty e^{-pt} \cdot 0\, dt$$

$$= \int_0^\pi e^{-pt} \sin t\, dt$$

$$= I_1 \qquad \text{—— (1)}$$

For I_1

$$I_1 = \int_0^\pi e^{-pt} \sin t\, dt$$

$$= \left[\sin t \, \frac{e^{-pt}}{-p}\right]_0^\pi - \int_0^\pi \cos t \cdot \frac{e^{-pt}}{-p}\, dt$$

$$= -\frac{1}{p}\left[0 - 0\right] + \frac{1}{p}\int_0^\pi e^{-pt} \cos t\, dt$$

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$$I_1 = \frac{1}{p}\int_0^{\pi} e^{-pt}\cos t\, dt$$

$$= \frac{1}{p}\left[\cos t \cdot \frac{e^{-pt}}{-p}\right]_0^{\pi} - \frac{1}{p}\int_0^{\pi}(-\sin t)\frac{e^{-pt}}{-p}\,dt$$

$$= \frac{1}{p}\left[(-1)\frac{e^{-p\pi}}{-p} + \frac{1}{p}\right] - \frac{1}{p^2}\int_0^{\pi}\sin t\, e^{-pt}\,dt$$

$$= \frac{e^{-p\pi}}{p^2} + \frac{1}{p^2} - \frac{1}{p^2} I_1 \qquad \{\text{By } ①\}$$

$$\Rightarrow I_1\left(1+\frac{1}{p^2}\right) = \frac{e^{-p\pi}+1}{p^2}$$

$$\Rightarrow I_1\left(\frac{p^2+1}{\cancel{p^2}}\right) = \frac{e^{-p\pi}+1}{\cancel{p^2}}$$

$$\Rightarrow I_1 = \frac{e^{-p\pi}+1}{p^2+1}$$

$\therefore$ ① becomes

$$L\{f(t)\} = \frac{e^{-p\pi}+1}{p^2+1} \quad \text{Ans.}$$

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5) Find $L\{\sin\sqrt{t}\}$

Ans:- We have,

$$L\{\sin\sqrt{t}\} = L\left\{\sqrt{t} - \frac{(\sqrt{t})^3}{\lfloor 3} + \frac{(\sqrt{t})^5}{\lfloor 5} - \cdots\right\}$$

$$= L\left\{t^{1/2} - \frac{t^{3/2}}{\lfloor 3} + \frac{t^{5/2}}{\lfloor 5} - \cdots\right\}$$

$$= L\{t^{1/2}\} - \frac{1}{\lfloor 3}L\{t^{3/2}\} + \frac{1}{\lfloor 5}L\{t^{5/2}\} - \cdots$$

$$= \frac{\Gamma(1+\frac{1}{2})}{p^{\frac{1}{2}+1}} - \frac{1}{\lfloor 3}\frac{\Gamma(1+\frac{3}{2})}{p^{\frac{3}{2}+1}} + \frac{1}{\lfloor 5}\frac{\Gamma(1+\frac{5}{2})}{p^{\frac{5}{2}+1}} - \cdots$$