

Boolean Functions.

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Let $B(+, \cdot, ')$ be a Boolean algebra and $\alpha(x_1, x_2, x_3, \dots, x_n)$ be a Boolean expression in n -variables x_1, x_2, \dots, x_n . Then we can determine a Boolean function $\alpha^B[x_1, x_2, \dots, x_n]$ as follows. For each $(a_1, a_2, \dots, a_n) \in B^n$, $\alpha^B[a_1, a_2, \dots, a_n]$ is an element of B obtained by assigning the values a_1, a_2, \dots, a_n to x_1, x_2, \dots, x_n respectively.

Thus "Any expression which is combination of a finite set of symbols i.e. a combination of constant and variables by operation $+$ & \cdot is called Boolean function."

Examples

(i) $f(x, y) = x + y'$

(ii) $f(x, y, z) = (a' + b)'c + ab'x + ac'y$

Minimal Boolean function

In case of n variables, a minimal function is product of all n variables (complement may occur).

For example (i) For two variables x, y , minimal functions are $x \cdot y$, $x' \cdot y$, $x \cdot y'$ and $x' \cdot y'$.

(ii) For three variables x, y & z , minimal functions are xyz , xyz' , $xy'z$, $x'y'z$, $xy'z'$, $x'yz'$, $x'y'z$, $x'y'z'$.

Bool's Theorem: There are 2^n minimal Boolean function in n variables.

Normal forms:

A Boolean function of n variable is said to be in normal form if it consists all n variables.

Types of normal form:

- (i) Disjunctive Normal form.
- (ii) Conjunctive Normal form.
- (iii) Complete Disjunctive Normal form.
- (iv) Complete Conjunctive Normal form.

Disjunctive Normal Form (DN form) or (DNF)

A Boolean function is said to be in conjunctive normal form in n variables x_1, x_2, \dots, x_n ; $n > 0$ if the expression is in the form of a sum of terms in which each term is of the type $f_1(x_1) \cdot f_2(x_2) \dots f_n(x_n)$

where $f_i(x_i) = x_i$ or x_i' $\forall i=1, 2, 3, \dots, n$,
and no two terms are same.

Ex $x + x'$ is DN form in one variable.
 $xy + x'y + x'y'$ is DN form in 2 variables.
 $xyz' + xy'z + xyz$ is DN form in 3 variables.

Th^m Every Boolean expression which contains no constant is equivalent to an expression in D.N. form.

(Proof is simple)

EXAMPLES

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- ① Express following Boolean expression in 3 variables $(x+y+z) \cdot (xy+x'z)'$ in D.N. form.

$$\begin{aligned}
 & (x+y+z) \cdot (xy+x'z)' \\
 &= (x+y+z) \cdot \{(xy)' \cdot (x'z)'\} \\
 &= (x+y+z) \cdot \{(x'+y') \cdot (x+z')\} \\
 &= (x+y+z) \cdot (x'x + x'z' + y'x + y'z') \\
 &= (x+y+z) \cdot (0 + x'z' + xy' + y'z') \\
 &= (x+y+z) \cdot (x'z' + xy' + y'z') \\
 &= xx'z' + xxy' + xy'z' + yx'z' + yy'z' + zzx'z' + zxy' + zyz' \\
 &= 0 + xy' + xy'z' + x'y'z' + 0 + 0 + 0 + xy'z + 0 \\
 &= xy'(z+z') + x'y'z' + x'y'z \\
 &= xy'z + xy'z' + x'y'z' + x'y'z + x'y'z \\
 &= (xy'z + xy'z') + (x'y'z' + x'y'z) + x'y'z' \\
 &= xy'z + xy'z' + x'y'z' \quad (\text{Required D.N form})
 \end{aligned}$$

- ② Express in D.N form $f(x,y,z) = [x+y' + (y+z)']' + yz$.

$$\begin{aligned}
 f(x,y,z) &= [x+y' + (y+z)']' + yz \\
 &= [x+y' \cdot 1 + y' \cdot z']' + yz \\
 &= [x+y'(1+z')] + yz \\
 &= [x+y' \cdot 1] + yz \\
 &= [x+y'] + yz \\
 &= x' \cdot y + yz
 \end{aligned}$$

$$= x'y(z+z') + yz(x+x')$$

$$= x'yz + x'yz' + xyz + x'yz$$

$$= x'yz + x'yz' + xyz$$

(Required D.N. form)

Complete Disjunctive Normal form:

A D.N. form in n variables which contains 2^n terms is called complete D.N. form.

① Complete D.N. form in two variables is

$$x.y + x'.y + x.y' + x'.y'$$

② Complete D.N. form in three variables is

$$xyz + xyz' + xy'z + x'yz + xy'z' + x'y'z' + x'yz' + x'y'z'$$

Thm A complete D.N. form is identically 1.

Proof Let f be any complete D.N. form in n variables x_1, x_2, \dots, x_n . We have coefficient of x_i & x_i' are same which are in terms of remaining $(n-1)$ variables.

Then $f = f_1(x_i + x_i') = f_1 \cdot 1 = f_1$, where f_1 is complete D.N. form in remaining $(n-1)$ variables containing 2^{n-1} terms. Repeating the process n -times we can eliminate all the variables and the expression reduces to 1.

(Proved)