Linear Difference Equations With n independent variables

Let the linear equation be

$$P_1 p_1 + P_2 p_2 + P_3 p_3 + \cdots + P_n p_n = R \quad \text{——— (1)}$$

Where $p_i = \dfrac{\partial z}{\partial x_i}$ $\quad i = 1, 2, 3, \ldots n.$

and $P_1, P_2, \cdots P_n$ and R are functions of $x_1, x_2, x_3 \ldots x_n$ and z

Then general integral of (1) is

$$f(u_1, u_2, \cdots u_n) = 0 \quad \text{——— (2)}$$

Where $u_1 = \text{const.}$ $u_2 = \text{const.}$ $\ldots$ $u_n = \text{const}$ are n-independent solutions subsidiary equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \frac{dx_3}{P_3} = \cdots = \frac{dx_n}{P_n} = \frac{dz}{R}. \quad \text{——— (3)}$$

SOME EXAMPLES

Example (1) Solve $p + q = \dfrac{z}{a}$

Sol^n Given differential equation is

$$p + q = z/a \quad \text{——— (1)}$$

Here $P = 1$ $Q = 1$ & $R = z/a$

So Lagrange's subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{z/a} \quad \text{——— (2)}$$

Taking first two terms of (2) we get

$$dx = dy$$

On integrating $x = y + c_1$

$$\Rightarrow x - y = c_1 \quad \text{——— (3)}$$

Again taking Last two terms of eqn ②

$$\frac{dy}{1} = \frac{dz}{z/a}$$

$$\Rightarrow \frac{dy}{a} = \frac{dz}{z}$$

On integrating

$$\Rightarrow \frac{y}{a} = \log z - \log c_2$$

$$\Rightarrow \frac{y}{a} = \log z/c_2$$

$$\Rightarrow e^{y/a} = z/c_2$$

$$\Rightarrow c_2 = z/e^{y/a} \quad \text{——— (4)}$$

Hence general integral is

$$\frac{z}{e^{y/a}} = f(x-y)$$

ie $z = e^{y/a} f(x-y)$

Ans

Example ② Solve $pz - qz = z^2 + (x+y)^2$

Solⁿ Here given differential equation is

$$pz - qz = z^2 + (x+y)^2 \quad \text{——— (1)}$$

So $P = z$ $Q = -z$ and $R = z^2 + (x+y)^2$

Lagrange's Subsidiary equation are

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2} \quad \text{——— (2)}$$

Taking first two terms of eqn ①

we get $\frac{dx}{z} = \frac{dy}{-z}$

$$\Rightarrow dx = -dy$$

On integrating $\Rightarrow x = -y + c_1$

$$\Rightarrow x + y = c_1 \quad \text{——— (3)}$$

Taking first and 3rd terms of (2) we have.

$$\frac{dx}{z} = \frac{dz}{z^2+(x+y)^2}$$

$$\Rightarrow \frac{dx}{z} = \frac{dz}{z^2+c_1^2} \quad \text{(From (3))}$$

$$\Rightarrow dx = \frac{z\,dz}{z^2+c_1^2}$$

$$\Rightarrow 2dx = \frac{2z\,dz}{z^2+c_1^2}$$

On integrating

$$2x = \log(z^2+c_1^2) + c_2$$

$$\Rightarrow 2x = \log\{z^2+(x+y)^2\} + c_2$$

$$\Rightarrow 2x - \log\{z^2+(x+y)^2\} = c_2$$

Hence general integral is

$$f\left(x+y,\ 2x-\log\{z^2+(x+y)^2\}\right) = 0.$$

Example (3) $z - xp - yq = a\sqrt{x^2+y^2+z^2}$

Sol^n The given differential equation can be written

as $xp + yq = z - a\sqrt{x^2+y^2+z^2}$ —— (1)

The subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z-a\sqrt{x^2+y^2+z^2}} \quad \text{—— (2)}$$

From first and second terms of (2)

$$\frac{dx}{x} = \frac{dy}{y}$$

On integrating

$$\log x = \log y + \log c_1$$

$$\Rightarrow \log x = \log c_1 y$$

$$\Rightarrow x = c_1 y$$

$$\Rightarrow x/y = c_1 \quad \text{---} (3)$$

Also

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - a\sqrt{x^2+y^2+z^2}} = \frac{x\,dx + y\,dy + z\,dz}{x^2+y^2+z^2 - az\sqrt{x^2+y^2+z^2}}$$

From 3rd and 4th terms we have

$$\frac{dx}{x} = \frac{dz}{z - a\sqrt{x^2+y^2+z^2}} = \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2) - a\sqrt{x^2+y^2+z^2}} \quad \text{---} (4)$$

Now putting $x^2+y^2+z^2 = u^2$

$$\Rightarrow 2x\,dx + 2y\,dy + 2z\,dz = 2u\,du$$

$$\Rightarrow x\,dx + y\,dy + z\,dz = u\,du.$$

Equation (4) becomes

$$\frac{dx}{x} = \frac{dz}{z - au} = \frac{u\,du}{u^2 - auz}$$

$$\Rightarrow \frac{dx}{x} = \frac{dz}{z - au} = \frac{du}{u - az} = \frac{du + dz}{(1-a)(u+z)}$$

ie $$\frac{dx}{x} = \frac{du + dz}{(1-a)(u+z)}$$

$$\Rightarrow (1-a)\frac{dx}{x} = \frac{d(u+z)}{(u+z)}$$

On integrating

$$(1-a)\log x = \log(u+z) + \log c_2$$

$$\Rightarrow \log x^{1-a} = \log c_2(u+z)$$

$$\Rightarrow x^{1-a} = c_2(u+z)$$

$$\Rightarrow \frac{x^{1-a}}{u+z} = c_2$$

ie $$\frac{x^{1-a}}{z + \sqrt{x^2+y^2+z^2}} = c_2 \quad \text{---} (5)$$

General soln is

$$\frac{x^{1-a}}{z + \sqrt{x^2+y^2+z^2}} = f(x/y)$$

Ans.