

SOME MORE EXAMPLES

Example ① Solve $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$

Solⁿ

Given ^{partial} differential equation is

$$x(y^2+z)p - y(x^2+z)q = z(x^2-y^2) \quad \text{--- ①}$$

Lagrange's subsidiary equations are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} \quad \text{--- ②}$$

Using multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, we get

$$\text{Each fraction of ②} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\Rightarrow \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating $\Rightarrow \log x + \log y + \log z = \log C_1$

$$\Rightarrow \log(xyz) = \log C_1$$

$$\Rightarrow xyz = C_1 \quad \text{--- ③}$$

Also using multipliers $x, y, -1$, we get.

$$\text{Each fraction of ②} = \frac{xdx + ydy - dz}{0}$$

$$\Rightarrow xdx + ydy - dz = 0$$

Integrating $\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} - z = \frac{C_2}{2}$

$$\Rightarrow x^2 + y^2 - 2z = C_2 \quad \text{--- ④}$$

Hence general solution is

$$f(xyz, x^2+y^2-2z) = 0$$

Ans

Example ② Solve $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} + t \frac{\partial z}{\partial t} = az + \frac{xy}{t}$.

Solⁿ

Given ^{partial} differential equation is

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} + t \frac{\partial z}{\partial t} = az + \frac{xy}{t} \quad \text{--- (1)}$$

The subsidiary equations are,

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{t} = \frac{dz}{az + \frac{xy}{t}} \quad \text{--- (2)}$$

From first and second terms of (2)

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\Rightarrow \log x = \log y + \log C_1$$

$$\Rightarrow \log x = \log C_1 y$$

$$\Rightarrow x = C_1 y$$

$$\Rightarrow \frac{x}{y} = C_1 \quad \text{--- (3)}$$

From first and third terms of (2)

$$\frac{dx}{x} = \frac{dt}{t}$$

$$\Rightarrow \log x = \log t + \log C_2$$

$$\Rightarrow \log x = \log C_2 t$$

$$\Rightarrow x = C_2 t$$

$$\Rightarrow \frac{x}{t} = C_2 \quad \text{--- (4)}$$

From first and fourth terms of (2)

$$\frac{dx}{x} = \frac{dz}{az + \frac{xy}{t}}$$

$$\Rightarrow x \frac{dz}{dx} = az + \frac{xy}{t}$$

$$\Rightarrow \frac{dz}{dx} = a \frac{z}{x} + \frac{y}{t}$$

$$\Rightarrow \frac{dz}{dx} = a \frac{z}{x} + \frac{c_2}{c_1} \quad \left\{ \because \frac{c_2}{c_1} = \frac{y}{t} \text{ From (3) \& (4)} \right\}$$

$$\Rightarrow \frac{dz}{dx} - a \frac{z}{x} = \frac{c_2}{c_1} \quad \text{--- (5)}$$

Which is linear differential equation in z .

$$\text{Integrating factor (IF)} = e^{-\int \frac{a}{x} dx}$$

$$= e^{-a \log x}$$

$$= \frac{1}{x^a}$$

Solution of eqn (5) is

$$z \cdot \frac{1}{x^a} = \frac{c_2}{c_1} \int \frac{1}{x^a} dx$$

$$\Rightarrow \frac{z}{x^a} = \frac{c_2}{c_1} \cdot \frac{x^{1-a}}{1-a} + c_3$$

$$\Rightarrow \frac{z}{x^a} = \frac{y}{t} \cdot \frac{x}{(1-a)x^a} + c_3$$

$$\Rightarrow \frac{z}{x^a} - \frac{y}{t} \cdot \frac{x}{(1-a)x^a} = c_3$$

$$\Rightarrow \frac{1}{x^a} \left\{ z - \frac{xy}{(1-a)t} \right\} = c_3 \quad \text{--- (6)}$$

Hence general integral is

$$f\left(\frac{x}{y}, \frac{x}{t}, \frac{1}{x^a} \left\{ z - \frac{xy}{(1-a)t} \right\}\right) = 0 \quad \text{Ans}$$

Example (3) solve $x_2 x_3 p_1 + x_3 x_1 p_2 + x_1 x_2 p_3 + x_1 x_2 x_3 = 0$

Solⁿ

Given partial differential eqn is

$$x_2 x_3 p_1 + x_3 x_1 p_2 + x_1 x_2 p_3 + x_1 x_2 x_3 = 0 \quad \text{--- (1)}$$

Lagrange's subsidiary equations are

$$\frac{dx_1}{x_2 x_3} = \frac{dx_2}{x_3 x_1} = \frac{dx_3}{x_1 x_2} = \frac{dz}{-x_1 x_2 x_3} \quad \text{--- (2)}$$

From first and end terms of (2)

$$\frac{dx_1}{x_2 x_3} = \frac{dx_2}{x_3 x_1}$$

$$\Rightarrow x_1 dx_1 = x_2 dx_2$$

$$\Rightarrow \frac{x_1^2}{2} = \frac{x_2^2}{2} + \frac{C_1}{2}$$

$$\Rightarrow x_1^2 = x_2^2 + C_1$$

$$\Rightarrow x_1^2 - x_2^2 = C_1 \text{ ————— (3)}$$

From first and 3rd terms of (2)

$$\frac{dx_1}{x_2 x_3} = \frac{dx_3}{x_1 x_2}$$

$$\Rightarrow x_1 dx_1 = x_3 dx_3$$

$$\Rightarrow \frac{x_1^2}{2} = \frac{x_3^2}{2} + \frac{C_2}{2}$$

$$\Rightarrow x_1^2 = x_3^2 + C_2$$

$$\Rightarrow x_1^2 - x_3^2 = C_2 \text{ ————— (4)}$$

From first and fourth terms of eqn (2)

$$\frac{dx_1}{x_2 x_3} = \frac{dz}{-x_1 x_2 x_3}$$

$$\Rightarrow x_1 dx_1 = -dz$$

$$\Rightarrow \frac{x_1^2}{2} = -z + \frac{C_3}{2}$$

$$\Rightarrow x_1^2 = -2z + C_3$$

$$\Rightarrow x_1^2 + 2z = C_3 \text{ ————— (5)}$$

General integral is

$$f(x_1^2 - x_2^2, x_1^2 - x_3^2, x_1^2 + 2z) = 0$$

Ans