(4)

15. **Antisymmetric tensor:** A tensor $A^{\mu\nu}$ is said to be antisymmetric or skew-symmetric if

$$A^{\mu\nu} = -A^{\nu\mu} \quad \text{——— (1)}$$

Property (1): An antisymmetric tensor remain unchanged by the tensor law of transformation.

Proof: If we show that $A'^{\mu\nu} = -A'^{\nu\mu}$ ——— (2)

By (1) $A^{\alpha\beta} = -A^{\beta\alpha}$ ——— (3)

Now
$$A'^{\mu\nu} = A^{\alpha\beta}\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}} = -A^{\beta\alpha}\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}} \text{ by (3)}$$
$$= -A^{\beta\alpha}\frac{\partial x'^{\nu}}{\partial x^{\beta}}\frac{\partial x'^{\mu}}{\partial x^{\alpha}} = -A'^{\nu\mu}$$
Proved

Property (2): An anti symmetric tensor $A^{\mu\nu}$ has $\frac{4(4-1)}{2}$ independent components.

$A^{\mu\nu}$ has 4^2 components which are given below:

$$\begin{matrix} A^{11} & A^{12} & A^{13} & A^{14} \\ A^{21} & A^{22} & A^{23} & A^{24} \\ A^{31} & A^{32} & A^{33} & A^{34} \\ A^{41} & A^{42} & A^{43} & A^{44}. \end{matrix}$$

Putting $\mu = \nu$ in (1), we have $A^{\mu\mu} = -A^{\nu\nu} \Rightarrow A^{\mu\mu} = 0$

Hence no. of independent components corresponding to a repeated suffix is 0.

Number of independent components of $A^{\mu\nu}$ corresponding to distinct suffices is 4^2-4. Due to antisymmetric property this number is reduced to $\frac{4^2-4}{2}$.

Total no. of independent components of $A^{\mu\nu}$
$$= \frac{4^2-4}{2} + 0 = \frac{4(4-1)}{2} = 6.$$

Note 1: A skew symmetric tensor $A^{\mu\nu}$ in n-dimension has $\frac{n(n-1)}{2}$ independent components.

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Note 2: A tensor $A_{\mu\nu\sigma}$ is said to be antisymmetric in suffices μ and ν if $A_{\mu\nu\sigma} = -A_{\nu\mu\sigma}$.
This tensor has $\frac{n(n-1)}{2} \cdot n = \frac{n^2}{2}(n-1)$ independent components.

Note 3: A tensor $A_{\mu\nu\sigma}$ is said to be skew-symmetric in suffices μ, ν and σ if

$$A_{\mu\nu\sigma} = -A_{\nu\mu\sigma},\quad A_{\mu\nu\sigma} = -A_{\mu\sigma\nu},\quad A_{\mu\nu\sigma} = -A_{\sigma\nu\mu}.$$

This tensor has ${}^nC_3 = \frac{n(n-1)(n-2)}{6}$ independent components in n-dimensions.

Theorem 1: Show that $\frac{\partial\phi}{\partial x^\alpha}$ is a covariant vector. Also show that dx^α is a contravariant vector.

Proof: Consider the transformation $x^i \to x'^i$

(i) $$\frac{\partial\phi}{\partial x'^\mu} = \frac{\partial\phi}{\partial x^\alpha} \cdot \frac{\partial x^\alpha}{\partial x'^\mu}$$

It is of the type $A'_\mu = A_\alpha \frac{\partial x^\alpha}{\partial x'_\mu}$

This proves that $\frac{\partial\phi}{\partial x^\alpha}$ is a covariant vector.

(ii) Evidently $dx'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha} dx^\alpha$.

Writing $dx^\mu = A^\mu$, then last becomes

$$A'^\mu = A^\alpha \frac{\partial x'^\mu}{\partial x^\alpha}.$$

This shows that A^α i.e. dx^α is contravariant vector.

Proved

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