Sabyasachi 17/4/2020

Differential Equation for UG Sem III

Paper cc-5

CLAIRAUT'S EQUATION

The equation of the form $y = px + f(p)$ is called Clairaut's equation, where $p = \frac{dy}{dx}$.

W.R. Diffing the given equation w.r.t. x. we will get one factor $\frac{dp}{dx} = 0$, After integrating it we will get $p = c$. Then eliminating p from the equation $y = px + f(p)$, we have $y = cx + f(c)$ is called the general solution or complete solution.

Second solution, also we will get called Singular solution.

If the complete solution is a quadratic Equation in c. Then find its descriminant (D). Then $D = 0$, is called the singular solution.

If the complete solution is not a quadratic Equation in c. Then differentiate it w.r.t. "c", treating x and y as constant for some time. From this we get the value of c. Then eliminating c from the complete solution, we will get the singular solution.

Now we are going to solve some problems.

Problem ① Solve the differential equation $y = px + a\sqrt{1+p^2}$ and also find the singular solution.

Solution: given equation is

$y = px + a\sqrt{1+p^2}$. It is a Clairaut's equation

Differentiating it w.r.t. x, we have

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + a\cdot\frac{1}{2\sqrt{1+p^2}} \times 2p\frac{dp}{dx}.$$

$$\Rightarrow p - p = \frac{dp}{dx}\left[x + \frac{ap}{\sqrt{1+p^2}}\right]$$

$$\Rightarrow \frac{dp}{dx}\left[x + \frac{ap}{\sqrt{1+p^2}}\right] = 0$$

Either $\frac{dp}{dx} = 0$, Integrating $p = c$

Eliminiting p form the given equation. The complete solution is

$$y = cx + a\sqrt{1+c^2} \longrightarrow (1)$$

For Singular Solution

$$y - cx = a\sqrt{1+c^2}, \text{ Squaring}$$

$$(y - cx)^2 = a^2(1+c^2)$$

$$\Rightarrow y^2 - 2cxy + c^2x^2 = a^2 + a^2c^2$$

$$\Rightarrow c^2x^2 - a^2c^2 - 2xyc + y^2 - a^2 = 0$$

$$\Rightarrow (x^2 - a^2)c^2 - 2xy.c + (y^2 - a^2) = 0$$

It is a quadratic Equation in c.

$$c\text{- Descriminant} = (-2xy)^2 - 4(x^2-a^2)(y^2-a^2)$$

$$= 4\left[x^2y^2 - (x^2-a^2)(y^2-a^2)\right]$$

$\therefore$ Singular solution is

$$4\left[x^2y^2 - (x^2y^2 - a^2x^2 - a^2y^2 + a^4)\right] = 0$$

$$\Rightarrow \cancel{x^2y^2} - \cancel{x^2y^2} + a^2x^2 + a^2y^2 - a^4 = 0$$

$$\Rightarrow a^2(x^2 + y^2 - a^2) = 0$$

$$\Rightarrow (x^2 + y^2 - a^2) = 0 \quad (\because a^2 \neq 0)$$

Ans

Problem (2) Solve the differential Equation $y = px + \sin^{-1} p$. Also find its singular solution.

Ans: Given Equation is $y = px + \sin^{-1} p$, It is a Clairaut's equation.

Diff*ing* it w.r.t. x. We have

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{1}{\sqrt{1-p^2}}\frac{dp}{dx}$$

$$\Rightarrow p - p = \frac{dp}{dx}\left[x + \frac{1}{\sqrt{1-p^2}}\right]$$

$$\Rightarrow \frac{dp}{dx}\left[x + \frac{1}{\sqrt{1-p^2}}\right] = 0$$

Either $\frac{dp}{dx} = 0 \Rightarrow$ Integrating, we have $p = c$

Eliminating p from the given Equation, The complete solution is $y = cx + \sin^{-1} c$ (1) Ans

For singular solution The equation (1) is not a quadratic Equation in c.

Now differentiating (1) w.r.t. c treating x and y as constant for some time.

$$0 = 1\cdot x + \frac{1}{\sqrt{1-c^2}} \Rightarrow -x = \frac{1}{\sqrt{1-c^2}}, \text{ squar}$$

$$\Rightarrow x^2 = \frac{1}{1-c^2} \Rightarrow 1 - c^2 = \frac{1}{x^2}$$

$$\Rightarrow c^2 = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$$

$$\Rightarrow c = \frac{\sqrt{x^2-1}}{x}$$

Eliminating c from eq (1), The singular solut

is $y = \frac{\sqrt{x^2-1}}{x}\cdot x + \sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right) = \sqrt{x^2-1} + \sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right)$

Ans.

Problem 3 Solve the differential Equation
$\cos y \cos px + \sin y \sin px - p = 0$, Also find it's singular solution.

Ans given Equation is

$$\cos y \cos px + \sin y \sin px - p = 0$$

$$\Rightarrow \cos y \cos px + \sin y \sin px = p$$

$$\Rightarrow \cos(y - px) = p$$

$$\Rightarrow y - px = \cos^{-1} p, \text{ or, } y = px + \cos^{-1} p,$$

It is a clairauts equation. diffing it w.r.t x

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{1}{\sqrt{1-p^2}}\frac{dp}{dx}.$$

Now solve it as problem 2.

Problem 4 Solve $y^2 + x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} = 4\left(\frac{dx}{dy}\right)^2$

Ans given Equation is written as

$$y^2 + x^2p^2 - 2xyp = 4\cdot\frac{1}{p^2}, \text{ where } p = \frac{dy}{dx}.$$

$$\Rightarrow (y - xp)^2 = \frac{4}{p^2}$$

$$\Rightarrow y - xp = \frac{2}{p} \Rightarrow y = px + \frac{2}{p} \quad \text{--- (1)}$$

It is calairauts Equation. Diffing it w.r.t. x.

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + 2\cdot\left(-\frac{1}{p^2}\right)\frac{dp}{dx}$$

$$\Rightarrow p - p = \frac{dp}{dx}\left(x - \frac{2}{p^2}\right)$$

$$\Rightarrow \frac{dp}{dx}\left(x - \frac{2}{p^2}\right) = 0$$

Either $\frac{dp}{dx} = 0$, Integrating $p = c$.
Eliminating p from equation (1), The Complete Solution is

$$y = cx + \frac{2}{c} \quad \text{--- (2) Ans}$$

For singular solution from equation (2) we have

$$y = \frac{c^2 x + 2}{c}$$

$\Rightarrow c^2 x + 2 = cy \Rightarrow c^2 x - cy + 2 = 0$

It is a quadratic equation in c.

c- Descriminant $= (-y)^2 - 4x.2$

$= y^2 - 8x$

$\therefore$ Singular solution is $y^2 - 8x = 0$ Ae

More about singular solution:

Let $f(x, y, p) = 0$ is a diff. Equation and its general solution is $\phi(x, y, c) = 0$. Both p and c descrimants contain the equation to the envelope, which is called singular solution.

More about locus of the envelopes

1. Tac locus: It will appear in p-Des. and not in c-Des. as c is different. The Tac locus does not give a solution.
2. Nodal locus: It will appear in c-descriminant only. Generally, nodal locus will not satisfy the diff. Equation
3. Cuspidal locus: It will appear in c-descrimenant as well as in p-descriminant. In general cuspidal locus will not satisfy the given diff. Equation.

Rule (i) The c-Dis. contains envelope once, nodal locus twice, cuspidal locus thrice. It does not contain tac locus.
In short, c-Descriminent $= EN^2C^3$.
(ii) The p-Dis. contains envelope once, tac locus twice and cuspidal locus once. It does not contain nodal locus.
In short, p-Des. $= ET^2C$.

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