Tensor calculus (3)

Theorem 2: Prove that there is no distinction between Contravariant and covariant vectors when we restrict ourselves to rectangular cartesian transformation of co-ordinates.

Proof: Let (x, y) be co-ordinates of a point P w.r.t. orthogonal cartesian axes X and Y. Let (x', y') be the co-ordinates of the same point P w.r.t. orthogonal cartesian axes X' and Y'. Let (l_1, m_1) and (l_2, m_2) be direction cosines of axes X' and Y' respectively. Then we have the relation

$$\left.\begin{aligned} x' &= l_1 x + m_1 y \\ y' &= l_2 x + m_2 y \end{aligned}\right\} \quad \text{——— (1)}$$

For which we have

$$\left.\begin{aligned} x &= l_1 x' + l_2 y' \\ y &= m_1 x' + m_2 y' \end{aligned}\right\} \quad \text{——— (2)}$$

Let $x^1 = x$, $x^2 = y$

Consider Contravariant transformation

$$A'^{\mu} = A^{\alpha} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} = A^1 \frac{\partial x'^{\mu}}{\partial x^1} + A^2 \frac{\partial x'^{\mu}}{\partial x^2}$$

$$\therefore \quad A'^1 = A^1 \frac{\partial x'^1}{\partial x^1} + A^2 \frac{\partial x'^1}{\partial x^2},$$

$$A'^2 = A^1 \frac{\partial x'^2}{\partial x^1} + A^2 \frac{\partial x'^2}{\partial x^2}$$

ie $A'^1 = A^1 \frac{\partial x'}{\partial x} + A^2 \frac{\partial x'}{\partial y}$, $A'^2 = A^1 \frac{\partial y'}{\partial x} + A^2 \frac{\partial y'}{\partial y}$

Writing these equations with the help of (1).

$$\left.\begin{aligned} A'^1 &= A^1 l_1 + A^2 m_1 \\ A'^2 &= A^1 l_2 + A^2 m_2 \end{aligned}\right\} \quad \text{——— (3)}$$

Consider Covariant transformation

$$A'_{\mu} = A_{\alpha} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} = A_1 \frac{\partial x^1}{\partial x'^{\mu}} + A_2 \frac{\partial x^2}{\partial x'^{\mu}}$$

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$\therefore\ A'_1 = A_1 \dfrac{\partial x^1}{\partial x'^1} + A_2 \dfrac{\partial x^2}{\partial x'^1}$, $A'_2 = A_1 \dfrac{\partial x^1}{\partial x'^2} + A_2 \dfrac{\partial x^2}{\partial x'^2}$

ie $A'_1 = A_1 \dfrac{\partial x}{\partial x'} + A_2 \dfrac{\partial y}{\partial x'}$, $A'_2 = A_1 \dfrac{\partial x}{\partial y'} + A_2 \dfrac{\partial y}{\partial y'}$

Writing these with the help of (2)

$$\left.\begin{aligned} A'_1 &= A_1 l_1 + A_2 m_1 \\ A'_2 &= A_1 l_2 + A_2 m_2 \end{aligned}\right\} \text{——— (4)}$$

Comparing (3) and (4), the required result follows.

Theorem (3): Prove that the laws of transformation of vectors possess group property.

Proof: (i) Let A^α be a covariant vector. Let us consider the transformation of coordinates

$$x^\mu \longrightarrow x'^\mu \longrightarrow x''^\mu$$
$$\text{(i)} \longrightarrow \text{(ii)} \longrightarrow \text{(iii)}$$
$$A^\mu \qquad A'^\mu \qquad A''^\mu$$

In case of (i) $\longrightarrow$ (ii), we have $A'^\alpha = A^\beta \dfrac{\partial x'^\alpha}{\partial x^\beta}$ — (1)

In case of (ii) $\longrightarrow$ (iii), we have $A''^\mu = A'^\alpha \dfrac{\partial x''^\mu}{\partial x'^\alpha}$

$$\therefore\ A''^\mu = A^\beta \frac{\partial x'^\alpha}{\partial x^\beta} \cdot \frac{\partial x''^\mu}{\partial x'^\alpha} \quad \text{by (1)}$$

$$\therefore\ A''^\mu = A^\beta \frac{\partial x''^\mu}{\partial x^\beta}.$$

From this it follows that if we make direct transformation from (i) to (iii), we get the same law of transformation. This property is expressed as follows: Contravariant vector of transformation possesses group property.

(ii) Let A_α be a covariant vector.

In case of (i) $\longrightarrow$ (ii), $A'_\alpha = A_\beta \dfrac{\partial x^\beta}{\partial x'^\alpha}$ — (2)

In case of (ii) $\longrightarrow$ (iii), $A''_\mu = A'_\alpha \frac{\partial x'^\alpha}{\partial x''^\mu}$

$$\therefore A''_\mu = A_\beta \frac{\partial x^\beta}{\partial x'^\alpha} \cdot \frac{\partial x'^\alpha}{\partial x''^\mu} = A_\beta \frac{\partial x^\beta}{\partial x''^\mu} \quad \text{by (2)}$$

From this it follows that if we make direct transformation from (i) to (iii), we get the same law of transformation so that covariant vector law of transformation possesses group property. Thus it follows that vector law of transformation possesses group property.

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