Putting, $\sqrt{pt} = x$

$\therefore \sqrt{p}\frac{1}{2\sqrt{t}} dt = dx$

$\Rightarrow \frac{dt}{\sqrt{t}} = \frac{2}{\sqrt{p}} dx$

$\therefore L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \int_0^\infty e^{-x^2} \frac{1}{\sqrt{\pi}} \frac{2}{\sqrt{p}} dx$

$= \frac{2}{\sqrt{\pi}\sqrt{p}} \int_0^\infty e^{-x^2} dx$

$= \frac{\cancel{2}}{\sqrt{\cancel{\pi}}\sqrt{p}} \cdot \frac{\sqrt{\cancel{\pi}}}{\cancel{2}}$

$= \frac{1}{\sqrt{p}}$ Ans.

—x—

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11/2018

First translation or Shifting theorem

If $L\{F(t)\} = f(p)$, when $p > 0$, then $L\{e^{at}F(t)\} = f(p-a)$, $p > a$.

ie; if $f(p)$ is Laplace transform of $F(t)$ then $f(p-a)$ is the Laplace transform of $e^{at}F(t)$.

Proof:- We have,

$f(p) = L\{F(t)\} = \int_0^\infty e^{-pt} F(t) dt$

$\therefore f(p-a) = \int_0^\infty e^{-(p-a)t} F(t) dt$

$= \int_0^\infty e^{-pt} \cdot \{e^{at} F(t)\} dt$

$= L\{e^{at} F(t)\}$ proved

Problems

(1) Find $L\{t^3 e^{-3t}\}$

Ans:

$$L\{t^3\} = \frac{\lfloor 3}{p^{3+1}} = \frac{\lfloor 3}{p^4} = \bar{f}(p) \quad (\text{say})$$

∴ From shifting theorem

$$L\{t^3 e^{-3t}\} = \bar{f}(p+3)$$

$$= \frac{\lfloor 3}{(p+3)^4} \quad \text{Ans}$$

(2) Find $L\{e^{-2t}(3\cos 6t - 5\sin 6t)\}$

Ans:-

$$\therefore L\{3\cos 6t - 5\sin 6t\}$$

$$= 3L\{\cos 6t\} - 5L\{\sin 6t\}$$

$$= 3 \times \frac{p}{p^2+36} - 5\frac{6}{p^2+36}$$

$$= \frac{3p - 6\times 5}{p^2+36} = \frac{3p-30}{p^2+36} = \bar{f}(p) \quad (\text{say})$$

∴ From shifting theorem.

$$L\{(3\cos 6t - 5\sin 6t)e^{-2t}\} = \bar{f}(p+2)$$

$$= \frac{3(p+2)-30}{(p+2)^2+36}$$

$$= \frac{3p-24}{p^2+4p+40} \quad \text{Ans}$$

③ Find $L\{e^{-t}(3\sinh 2t - 5\cosh 2t)\}$

Ans:-

$L\{3\sinh 2t - 5\cosh 2t\}$

$= 3L\{\sinh 2t\} - 5L\{\cosh 2t\}$

$= 3\dfrac{2}{p^2-4} - 5\dfrac{p}{p^2-4}$

$= \dfrac{6-5p}{p^2-4} = f(p)$ (say)

$\therefore$ from shifting theorem

$L\{e^{-t}(3\sinh 2t - 5\cosh 2t)\} = f(p+1)$

$= \dfrac{6-5(p+1)}{(p+1)^2-4}$

$= \dfrac{6-5p-5}{p^2+2p+1-4}$

$= \dfrac{1-5p}{p^2+2p-3}$ Ans

—x—

④ Find $L\{e^{t}\sin^2 t\}$

Ans:-

$L\{\sin^2 t\} = L\left\{\dfrac{1-\cos 2t}{2}\right\} = \dfrac{1}{2}L\{1\} - \dfrac{1}{2}L\{\cos 2t\}$

$= \dfrac{1}{2}\dfrac{1}{p} - \dfrac{1}{2}\dfrac{p}{p^2+4} =$

$= \dfrac{1}{2}\left(\dfrac{1}{p} - \dfrac{p}{p^2+4}\right)$

$= \dfrac{1}{2}\dfrac{p^2+4-p^2}{p(p^2+4)}$

$= \dfrac{2}{p(p^2+4)} = f(p)$ (say)

$\therefore L\{e^{t}\sin^2 t\} = f(p-1)$ {From shifting theorem

$= \frac{2}{(p-1)\{(p-1)^2+4\}}$

$= \frac{2}{(p-1)(p^2-2p+5)}$ Ans.

—x—

(5) Find $L\left\{ e^{-at} \frac{t^{n-1}}{\lfloor n-1} \right\}$

Ans:-

$L\left\{ \frac{t^{n-1}}{\lfloor n-1} \right\} = \frac{1}{\lfloor n-1} L\{t^{n-1}\}$

$= \frac{1}{\cancel{\lfloor n-1}} \frac{\cancel{\lfloor n-1}}{p^{n-1+1}} = \frac{1}{p^n} = f(p)$, (say)

From shifting theorem,

$L\left\{ e^{-at} \frac{t^{n-1}}{\lfloor n-1} \right\} = f(p+a)$

$= \frac{1}{(p+a)^n}$ Ans.

—x—

(6) Find the Laplace transform of
$e^{-t}(3\sin 2t - 5\cosh 2t)$

Ans:-

$L\{3\sin 2t - 5\cosh 2t\} = 3\frac{2}{p^2+4} - 5\frac{p}{p^2-4}$

$= \frac{6}{p^2+4} - \frac{5p}{p^2-4} = f(p)$ (say)

$\therefore L\{e^{-t}(3\sin 2t - 5\cosh 2t)\} = f(p+1)$

$= \frac{6}{(p+1)^2+4} - \frac{5(p+1)}{(p+1)^2-4}$

$= \frac{6}{p^2+2p+5} - \frac{5(p+1)}{p^2+2p-3}$

(7) Find the laplace transform of the function $(t+3)^2 e^{-t}$.

Ans:-

$$\therefore L\{t+3\}^2 = L\{t^2+6t+9\}$$

$$= L\{t^2\}+6L\{t\}+9L\{1\}$$

$$= \frac{\lfloor 2}{p^3} + 6\frac{\lfloor 1}{p^2} + \frac{9}{p}$$

$$= \frac{2}{p^3} + \frac{6}{p^2} + \frac{9}{p} = f(p) \text{ (say)}$$

From shifting theorem.

$$L\{(t+3)^2 e^{-t}\} = f(p-1)$$

$$= \frac{2}{(p+1)^3} + \frac{6}{(p+1)^2} + \frac{9}{(p+1)}$$

$$= \frac{1}{(p+1)}\left[\frac{2}{(p+1)^2} + \frac{6}{(p+1)} + 9\right]$$

$$= \frac{1}{(p+1)}\left[\frac{2+6(p+1)+9(p+1)^2}{(p+1)^2}\right]$$

$$= \left[\frac{2+6p+6+9p^2+18p+9}{(p+1)^3}\right] = \frac{9p^2+24p+17}{(p+1)^3} \text{ Ans}$$

—x—

*

(8) If $L\{F(t)\} = f(p)$. Find $L\{F(t)\cos\omega t\}$.

Ans:- We have

$$L\{F(t)\cos\omega t\} = L\left\{F(t)\left(\frac{e^{i\omega t}+e^{-i\omega t}}{2}\right)\right\}$$

$$= \frac{1}{2}L\{e^{i\omega t}F(t)\} + \frac{1}{2}L\{e^{-i\omega t}F(t)\}$$

$$= \frac{1}{2}\left[f(p-i\omega) + f(p+i\omega)\right]$$

[By 1st shifting theorem]

—x—