Some problems for practice

Solve the following partial differential equation

1. $\cos(x+y)\cdot p + \sin(x+y)\cdot q = z$
2. $\frac{y^2 z}{x} p + zxq = y^2$
3. $(y^2+z^2-x^2)p - 2xyq + 2xz = 0$
4. $(y+zx)p - (x+yz)q = x^2-y^2$

NON-LINEAR PARTIAL DIFFERENTIAL EQUATION

(Some special types)

Type I: A partial differential equation in which only p and q are present.

ie eqn of the form $f(p,q)=0$ —— (1)

Complete Integral

Then complete integral is given by

$$z = ax + by + c \quad \text{—— (2)}$$

where a and b are connected by $f(a,b)=0$

Now from $f(a,b)=0$ we can find b interms of "a" as $b=\phi(a)$

Thus complete solution is

$$z = ax + \phi(a)y + c \quad \text{—— (3)}$$

General Integral

We have complete integral

$$z = ax + \phi(a)y + c$$

Putting $c=\psi(a)$ where ψ is arbitrary function we have

$$z = ax + \phi(a)y + \psi(a) \quad \text{—— (4)}$$

Partially Differentiating (4) w.r.to a we get

$$0 = x + \phi'(a)y + \psi'(a) \quad \text{—— (5)}$$

Eliminating a from (4) & (5) gives general solution.

Singular Integral

The complete integral is

$$z = ax + \phi(a)y + c$$

Partially differentiating it w.r.to a & w.r.to c we get

$$0 = x + \phi'(a)y$$

$$\& \quad 0 = 1$$

Since $0=1$ is not admissible, so no envelop is there

Hence ~~no~~ it has no singular solution.

Ex 1. Solve $p^2+q^2=1$.

Sol^n Given partial differential equation is

$$p^2+q^2=1 \quad \text{—— (1)}$$

It is clearly of type $f(p,q)=0$

It's complete solution is given by

$$z = ax + by + c \quad \text{—— (2)}$$

where $a^2+b^2=1$

$$\Rightarrow b = \sqrt{1-a^2}$$

Putting in (2) we get the complete integral as

$$\boxed{z = ax + \sqrt{1-a^2}\,y + c}$$

Putting $c = \psi(a)$ we get

$$z = ax + \sqrt{1-a^2}\,y + \psi(a) \quad \text{—— (3)}$$

Differentiating partially w.r.to a we get

$$0 = x + \frac{-a}{\sqrt{1-a^2}}\,y + \psi'(a) \quad \text{—— (4)}$$

The general solution is obtained by eliminating a from eqn (3) & (4)

Ans

Ex (2) Solve $p^2+q^2=3pq$

sol^n Given partial differential equation is

$$p^2+q^2=3pq \quad \text{--- (1)}$$

Which is of the form $f(p,q)=0$

So the complete integral is given by

$$z=ax+by+c \quad \text{--- (2)}$$

Where $a^2+b^2=3ab$

$$\Rightarrow b^2-3ab+a^2=0$$

$$\Rightarrow b=\frac{-(-3a)\pm\sqrt{9a^2-4a^2}}{2}$$

$$\Rightarrow b=\frac{3\pm\sqrt{5}}{2}a$$

Putting in (2) we get complete integral as

$$\boxed{z=ax+\left(\frac{3\pm\sqrt{5}}{2}\right)ay+c}$$

Now equation reducible to the form $f(p,q)=0$

Ex (3) Solve $x^2p^2+y^2q^2=z^2$

sol^n Given differential eq^n is

$$x^2p^2+y^2q^2=z^2$$

$$\Rightarrow \frac{x^2}{z^2}p^2+\frac{y^2}{z^2}q^2=1$$

ie $$\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2+\left(\frac{y}{z}\frac{\partial z}{\partial y}\right)=1. \quad \text{--- (1)}$$

Let us substitute

$$X=\log x \quad \Rightarrow dX=\frac{dx}{x}$$

$$Y=\log y \quad \Rightarrow dY=\frac{dy}{y}$$

$$Z=\log z \quad \Rightarrow dZ=\frac{dz}{z}$$

$$\Rightarrow \left(\frac{x}{z}\frac{\partial z}{\partial x}\right) = \frac{\partial Z}{\partial X} = P \text{ (say)}$$

$$\& \left(\frac{y}{z}\frac{\partial z}{\partial y}\right) = \frac{\partial Z}{\partial Y} = Q \text{ (say)}$$

Equation (1) becomes

$$P^2 + Q^2 = 1 \quad \text{——— (2)}$$

It's complete solution is given by

$$Z = aX + bY + c. \quad \text{——— (3)}$$

where $a^2 + b^2 = 1$

$$\Rightarrow b = \sqrt{1-a^2}$$

Putting in eqn (3) we get

$$Z = aX + \sqrt{1-a^2}\,Y + c$$

ie $\log z = a \log x + \sqrt{1-a^2} \log y + \log k$

(Putting $c = \log k$)

Also putting $a = \cos\alpha$

$$\Rightarrow \sqrt{1-a^2} = \sin\alpha$$

we get $\log z = \cos\alpha \cdot \log x + \sin\alpha \cdot \log y + \log k$

$$\Rightarrow \log z = \log\left(x^{\cos\alpha} \cdot y^{\sin\alpha} \cdot k\right)$$

$$\Rightarrow \boxed{z = k\, x^{\cos\alpha} \cdot y^{\sin\alpha}}$$

Ans