Some problems for Practice (Rivision)

① Solve $(y-x)(qy-px) = (p-q)^2$ [Hint: $x+y=u$ $xy=v$]

② Solve $p = e^q$

③ Solve $p^2 - q^2 = 4$

④ Solve $pq = x^m y^n z^l$ [Hint: Let $X = \frac{x^{m+1}}{m+1}$, $Y = \frac{y^{n+1}}{n+1}$, $Z = \frac{z^{l-1/2}}{l-1/2}$]

Type II Equation involving p, q and z ie equation of the form $f(z, p, q) = 0$ —— ①

Let us assume that $z = f(x+ay)$ is a trial solⁿ of eqn ①, where a is any arbitrary constant.

$$\therefore \; z = f(X) \text{ where } X = x + ay$$

$$\Rightarrow p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}\cdot 1 = \frac{dz}{dX}$$

$$\& \; q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial y} = \frac{\partial z}{\partial X}\cdot a = a\frac{dz}{dX}$$

Then eqn ① becomes

$$f\left(z, \frac{dz}{dX}, a\frac{dz}{dX}\right) = 0 \quad \text{—— ②}$$

Clearly eqn ② is ordinary differential eqn of first order. Integrating ② we get complete integral.

In complete integral $F(x, y, z, a, b) = 0$ if we put $b = \psi(a)$ where ψ is arbitrary function, then the general solution is obtained by eliminating a from $F = 0$ & $\frac{\partial F}{\partial a} = 0$

If $F(x,y,z,a,b)=0$ is complete integral then the solution obtained by eliminating $F=0$ $\frac{\partial F}{\partial a}=0$ & $\frac{\partial F}{\partial b}=0$ is called singular solution

Example 01 Find the complete solution of $p^2=zq$.

Sol^n: Given partial differential eqn. is

$$p^2=zq \quad \text{——— } (1)$$

Putting $z=f(x+ay)=f(X)$.

$$\Rightarrow p=\frac{dz}{dX} \quad \& \quad q=a\frac{dz}{dX}$$

Putting these values in (1) we get

$$\left(\frac{dz}{dX}\right)^2=za\frac{dz}{dX}$$

$$\Rightarrow \frac{dz}{dX}=za$$

$$\Rightarrow \frac{dz}{z}=a\,dX$$

On integrating we get

$$\log z=aX+\log b$$

$$\Rightarrow \log z-\log b=aX$$

$$\Rightarrow \log\frac{z}{b}=aX$$

$$\Rightarrow \frac{z}{b}=e^{aX}$$

$$\Rightarrow z=b\,e^{a(x+ay)}$$

Which is complete solution

Example ② Solve $9(p^2 z+q^2)=4$.

Soln. Given partial differential equation is

$$9(p^2 z+q^2)=4 \quad \text{--- (1)}$$

Putting $z=f(x+ay)=f(X)$

$$\Rightarrow \frac{dz}{dX}=p \quad \text{and} \quad a\frac{dz}{dX}=q$$

Putting these values in eqn (1) we get

$$9\left\{\left(\frac{dz}{dX}\right)^2 z+\left(a\frac{dz}{dX}\right)^2\right\}=4$$

$$\Rightarrow 9\left(\frac{dz}{dX}\right)^2(z+a^2)=4$$

$$\Rightarrow \left(\frac{dz}{dX}\right)^2=\frac{4}{9(z+a^2)}$$

$$\Rightarrow \frac{dz}{dX}=\frac{2}{3\sqrt{z+a^2}}$$

$$\Rightarrow 3\sqrt{z+a^2}\,dz=2dX$$

On integrating we get

$$2X=3\cdot\frac{2}{3}(z+a^2)^{3/2}+2b$$

$$\Rightarrow X=(z+a^2)^{3/2}+b$$

$$\Rightarrow x+ay=(z+a^2)^{3/2}+b$$

Which is complete solution.

Example ③. Solve $z^2(p^2+q^2+1)=4$

Sol^n Given partial differential equation is

$$z^2(p^2+q^2+1)=4 \quad \text{——— (1)}$$

Let $z=f(x+ay)=f(X)$

$$\Rightarrow \quad p=\frac{dz}{dX} \text{ and } q=a\frac{dz}{dX}$$

Putting these values in (1) we get

$$z^2\left\{\left(\frac{dz}{dX}\right)^2+\left(a\frac{dz}{dX}\right)^2+1\right\}=4$$

$$\Rightarrow \left(\frac{dz}{dX}\right)^2+a^2\left(\frac{dz}{dX}\right)^2+1=\frac{4}{z^2}$$

$$\Rightarrow \left(\frac{dz}{dX}\right)^2(1+a^2)=\frac{4}{z^2}-1$$

$$\Rightarrow \frac{dz}{dX}\sqrt{1+a^2}=\frac{\sqrt{4-z^2}}{z}$$

$$\Rightarrow \frac{\sqrt{1+a^2}}{\sqrt{4-z^2}}z\,dz=dX$$

On integrating

$$\sqrt{1+a^2}\int\frac{z\,dz}{\sqrt{4-z^2}}=\int dX+b$$

$$\Rightarrow -\sqrt{1+a^2}\sqrt{4-z^2}=X+b$$

$$\Rightarrow -\sqrt{1+a^2}\sqrt{4-z^2}=(x+ay+b)$$

On squaring

$$\Rightarrow (1+a^2)(4-z^2)=(x+ay+b)^2$$

Which is complete integral.