Some problems for Revision

Solve following partial differential eqn

① $pz = 1 + q^2$

② $p^2 = z^2(1 - pq)$

③ $p^3 + q^3 = 27z$

④ $p + q = z/a$

⑤ $p(z+p) + q = 0$

⑥ $z^2 = 1 + p^2 + q^2$

Type 3 Equation of the form $f(x,p) = F(y,q)$

As a trial solution let

$$f(x,p) = F(y,q) = a \quad \text{— ①}$$

From which we can obtain p and q as

$$p = f_1(x,a)$$

$$\& \quad q = f_2(y,a)$$

Now from $dz = p\,dx + q\,dy$

We get $dz = f_1(x,a)\,dx + f_2(y,a)\,dy$

On integrating we get

$$z = \int f_1(x,a)\,dx + \int f_2(y,a)\,dy + b$$

Which is complete integral of given p.d.e.

As discussed in previous two cases we can find the general integral & singular integral.

3) Now let us consider some examples

Ex ① Solve $\sqrt{p}+\sqrt{q}=2x$

Solⁿ Given $\sqrt{p}-2x=-\sqrt{q}$

So let $\sqrt{p}-2x=-\sqrt{q}=a$

$$\Rightarrow \sqrt{p}=a+2x$$

and $p=(a+2x)^2$

Similarly $-\sqrt{q}=a$

$$\Rightarrow q=a^2.$$

Putting these values in

$$dz=p\,dx+q\,dy$$

We get $dz=(a+2x)^2dx+a^2dy$

Which on integrating we get

$$z=\frac{(a+2x)^3}{6}+a^2y+b$$

Which is complete integral.

Ex ② Solve $yp=2yx+\log q$

Solⁿ Given $yp=2yx+\log q$

$$\Rightarrow y(p-2x)=\log q$$

$$\Rightarrow p-2x=\frac{1}{y}\log q$$

Let $p-2x=\frac{1}{y}\log q=a$

$$\Rightarrow p=a+2x \text{ and } q=e^{ay}$$

Putting values of p and q in

$$dz = p\,dx + q\,dy$$

we get $dz = (a+2x)dx + e^{ay}dy$

Which on integrating we get

$$z = ax + x^2 + \frac{e^{ay}}{a} + b$$

Which is the complete integral.

Ex ③ Solve $z^2(p^2+q^2) = x^2+y^2$

Sol^n Given $z^2(p^2+q^2) = x^2+y^2$ —— ①

Let $\frac{z^2}{2} = Z$

$$\Rightarrow z\,\partial z = \partial Z$$

$$\Rightarrow z\frac{\partial z}{\partial x} = \frac{\partial Z}{\partial x} = P \text{ (say)}$$

$$\& \quad z\frac{\partial z}{\partial y} = \frac{\partial Z}{\partial y} = Q \text{ (say)}$$

Eqn ① reduces to

$$P^2 + Q^2 = x^2 + y^2$$

$$\Rightarrow P^2 - x^2 = y^2 - Q^2$$

Let $P^2 - x^2 = y^2 - Q = a$

$$\Rightarrow P = \sqrt{a + x^2}$$

$$\& \quad Q = \sqrt{y^2 - a}$$

Putting value of P & Q in

$$dZ = P\,dx + Q\,dy$$

we get

$$dZ = \sqrt{a+x^2}\,dx + \sqrt{y^2-a}\,dy$$

Which on integrating we get

$$Z = \frac{x\sqrt{a+x^2}}{2} + \frac{a}{2}\log\left(x+\sqrt{a+x^2}\right) + \frac{y\sqrt{y^2-a}}{2} - \frac{a}{2}\log\left(y+\sqrt{y^2-a}\right) + b/2$$

Putting the value of Z

$$\frac{z^2}{2} = \frac{x\sqrt{a+x^2}}{2} + \frac{a}{2}\log\left(x+\sqrt{a+x^2}\right) + \frac{y\sqrt{y^2-a}}{2} - \frac{a}{2}\log\left(y+\sqrt{y^2-a}\right) + b/2$$

$$\Rightarrow z^2 = x\sqrt{a+x^2} + a\log\left(x+\sqrt{a+x^2}\right) + y\sqrt{y^2-a} - a\log\left(y+\sqrt{y^2-a}\right) + b$$

Which is required complete integral.

Some problems for practice

Solve.

(1) $(p^2+q^2) = (x^2+y^2)z$ [Hint put $2\sqrt{z} = Z$]

(2) $(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1$ [Hint put $x+y=u$ & $x-y=v$]

(3) $z(p^2-q^2) = x-y$ [Hint put $\frac{2}{3}z^{3/2} = Z$]

Type ④ Equation of the form $z = px + qy + f(p,q)$
(Analogous to Clairaut's form).

It's complete integral is
$$z = ax + by + f(a,b)$$
General integral and singular integral can be found as discussed earlier.

Ex ① Solve $z = px + qy + c\sqrt{1+p^2+q^2}$.

Solⁿ Give partial differential eqn is:
$$z = px + qy + c\sqrt{1+p^2+q^2} \quad \text{——} ① \quad (\text{type } 4)$$
So complete solⁿ can be obtained by putting $p = a$ & $q = b$ in ①

ie $$z = ax + by + c\sqrt{1+a^2+b^2} \quad \text{——} ②$$

Now to find singular solution, we differentiate partially eqn ② w.r.to a and b respectively we get
$$0 = x + \frac{ac}{\sqrt{1+a^2+b^2}} \quad \text{——} ③$$
$$\& \quad 0 = y + \frac{bc}{\sqrt{1+a^2+b^2}} \quad \text{——} ④$$
We want to eliminate a & b from ② ③ & ④

We have
$$x^2 + y^2 = \left(\frac{-ac}{\sqrt{1+a^2+b^2}}\right)^2 + \left(\frac{-bc}{\sqrt{1+a^2+b^2}}\right)^2$$
$$\Rightarrow x^2 + y^2 = \frac{(a^2+b^2)c^2}{1+a^2+b^2}$$

$$\Rightarrow c^2-(x^2+y^2)=c^2-\frac{(a^2+b^2)c^2}{1+a^2+b^2}$$

$$\Rightarrow c^2-(x^2+y^2)=\frac{c^2}{1+a^2+b^2}$$

$$\Rightarrow 1+a^2+b^2=\frac{c^2}{c^2-(x^2+y^2)} \quad \text{——— } (5)$$

From (3) $a=-x\dfrac{\sqrt{1+a^2+b^2}}{c}$

$$=-x\frac{1}{\sqrt{c^2-x^2-y^2}} \quad (\text{From } (5))$$

From (4) $b=-y\dfrac{\sqrt{1+a^2+b^2}}{c}$

$$=-y\frac{1}{\sqrt{c^2-x^2-y^2}}$$

Putting the values of a & b in (2) we get

$$z=-\frac{x^2}{\sqrt{c^2-x^2-y^2}}-\frac{y^2}{\sqrt{c^2-x^2-y^2}}+\frac{c^2}{\sqrt{c^2-x^2-y^2}}$$

$$\Rightarrow z=\frac{c^2-x^2-y^2}{\sqrt{c^2-x^2-y^2}}$$

$$\Rightarrow z=\sqrt{c^2-x^2-y^2} \quad \text{ie } x^2+y^2+z^2=c^2$$

is singular sol^n

Problems for Practice

(1) $z=px+qy-2p-3q$

(2) $z=px+qy-p^2q$

(3) $z=px+qy+pq$.