Second Translation or Shifting theorem

If $L\{F(t)\} = f(p)$ and $G(t) = \begin{cases} F(t-a) &, t>a \\ 0 &, t<a \end{cases}$

then $L\{G(t)\} = e^{-ap} f(p)$

Proof: By defination we have

$$L\{G(t)\} = \int_0^\infty e^{-pt} G(t)\,dt$$

$$= \int_0^a e^{-pt} \cdot 0 \cdot dt + \int_a^\infty e^{-pt} F(t-a)\,dt$$

$$= \int_a^\infty e^{-pt} F(t-a)\,dt$$

Putting $t-a = x$

$dt = dx$

when $t=a$ then $x=0$

" $t=\infty$ " $x=\infty$

$$\therefore L\{G(t)\} = \int_0^\infty e^{-p(a+x)} F(x)\,dx$$

$$= \int_0^\infty e^{-pa-px} F(x)\,dx$$

$$= \int_0^\infty e^{-px} \cdot e^{-pa} F(x)\,dx$$

$$= e^{-pa} \int_0^\infty e^{-px} F(x)\,dx$$

$$= e^{-pa} \int_0^\infty e^{-pt} F(t)\,dx \qquad \left[\because \int_a^b f(x)\,dx = \int_a^b f(t)\,dt\right.$$

$$= e^{-pa} L\{F(t)\}$$

$$= e^{-pa} f(p)$$

— x —

Problems

① Find $L\{G(t)\}$, where $G(t) = \begin{cases} e^{t-a}, & t>a \\ 0, & t<a \end{cases}$

Ans:- By 2nd shifting theorem, we have.

if $L\{F(t)\} = f(p)$

and $G(t) = \begin{cases} e^{t-a}, & t>a \\ 0, & t<a \end{cases}$

then $L\{G(t)\} = e^{-ap} f(p)$

Here $F(t) = e^{t}$

$$\therefore L\{e^t\} = \int_0^\infty e^{-pt} \cdot e^t \, dt$$

$$= \int_0^\infty e^{-(p-1)t} \, dt$$

$$= \left[\frac{e^{-(p-1)t}}{-(p-1)}\right]_0^\infty \quad \text{if } p>1$$

$$= \left[0 + \frac{1}{(p-1)}\right] \quad \text{if } p>1$$

$$= \frac{1}{p-1} \quad \text{if } p>1$$

$$= f(p)$$

and $G(t) = \begin{cases} e^{t-a}, & t>a \\ 0, & t<a \end{cases}$

$$\therefore L\{G(t)\} = e^{-ap} f(p)$$

$$= e^{-ap} \frac{1}{p-1} \quad \text{if } p>1.$$

② Find $L\{F(t)\}$ where

$$F(t) = \begin{cases} \cos(t - \frac{2}{3}\pi) & ; t > \frac{2}{3}\pi \\ 0 & ; t < \frac{2\pi}{3} \end{cases}$$

Ans:

Let $\phi(t) = \cos t$

$$\therefore F(t) = \begin{cases} \phi(t - \frac{2}{3}\pi) & ; t > \frac{2\pi}{3} \\ 0 & ; t < \frac{2\pi}{3} \end{cases}$$

We have,

$$L\{\phi(t)\} = L\{\cos t\}$$

$$= \frac{p}{p^2+1} = f(p) \quad (\text{say})$$

$\therefore$ From the second shifting theorem

$$L\{F(t)\} = e^{-ap} f(p)$$

$$= e^{(-2\pi/3)p} \times \frac{p}{p^2+1}$$

③ Find $L\{F(t)\}$ where

$$F(t) = \begin{cases} \sin(t - \frac{\pi}{3}) & , t > \frac{\pi}{3} \\ 0 & , t < \pi/3 \end{cases}$$

Ans:

Let $\phi(t) = \sin t$

$$\therefore F(t) = \begin{cases} \phi(t - \frac{\pi}{3}) & ; t > \frac{\pi}{3} \\ 0 & ; t < \pi/3 \end{cases}$$

We have,

$$L\{\phi(t)\} = L\{\sin t\} = \frac{1}{p^2+1} = f(p) \quad (\text{say})$$

$\therefore$ From 2nd shifting theorem, we have.

$$L\{F(t)\} = e^{-ap} f(p)$$

$$= e^{-\frac{\pi}{3}p} \frac{1}{p^2+1} \quad \text{Ans}$$

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(3) $F(t) = \begin{cases} \sin(t - \frac{2\pi}{3}) & , t > \frac{2\pi}{3} \\ 0 & , t < \frac{2\pi}{3} \end{cases}$

Ans: Let $\phi(t) = \sin t$

$$\therefore F(t) = \begin{cases} \phi(t - \frac{2\pi}{3}) & , t > \frac{2\pi}{3} \\ 0 & , t < \frac{2\pi}{3} \end{cases}$$

$$\therefore L\{\phi(t)\} = L\{\sin t\}$$

$$= \frac{1}{p^2+1}$$

$\therefore$ From 2nd shifting theorem we have,

$$L\{F(t)\} = e^{-ap} f(p)$$

$$= e^{-\frac{2\pi}{3}p} \frac{1}{p^2+1}$$

$$= \frac{e^{-\frac{2\pi}{3}p}}{p^2+1} \quad \text{Ans}$$

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