Tensor Calculus (7)

Definition: Inner product of two vectors: let A^α be contravariant vector and B_α is a covariant vector. The product $A^\alpha B_\alpha$ is called inner product or scalar product of vectors A^α and B_α. The product $A^\alpha B_\beta$ is called outer or open product of A^α and B_β.

Th (6) Prove that $A^\alpha B_\alpha$ is a scalar.

Proof:

$$A'^{\mu} B'_{\mu} = A^\alpha \frac{\partial x'^{\mu}}{\partial x^\alpha} B_\beta \frac{\partial x^\beta}{\partial x'^{\mu}}$$

$$= A^\alpha B_\beta \frac{\partial x'^{\mu}}{\partial x^\alpha} \cdot \frac{\partial x^\beta}{\partial x'^{\mu}} = A^\alpha B_\beta \frac{\partial x^\beta}{\partial x^\alpha}$$

$$= A^\alpha B_\beta \delta_\alpha^\beta = A^\alpha B_\alpha = A^\mu B_\mu$$

$$\therefore A'^{\mu} B'_{\mu} = A^\mu B_\mu .$$

This proves that $A^\mu B_\mu$ remains unchanged by tensor law of transformation and hence $A^\mu B_\mu$ is a scalar or invariant.

Definition: Multiplication of tensors: The product of two tensors is a tensor whose rank is the sum of rank of the two tensors.

e.g. If we multiply a tensor $A^{\mu_1 \mu_2 \ldots \mu_l}_{\nu_1 \nu_2 \ldots \nu_m}$ (which is covariant of order m and contravariant of order l) by a tensor $B^{\alpha_1 \alpha_2 \ldots \alpha_p}_{\beta_1 \beta_2 \ldots \beta_q}$ (which is covariant of order q and contravariant of order p) the product obtained is a tensor which is covariant of order $m+q$ and contravariant of order $l+p$. This product is called open product or outer product of two tensors.

Theorem (7): The product (open product) of two tensors is a tensor.

Proof: Let A^{μ}_{ν} and B_{σ} be any two tensors.

Let $C^{\mu}_{\nu\sigma} = A^{\mu}_{\nu} B_{\sigma}$ ——— (1)

If we show that $C^{\mu}_{\nu\sigma}$ is a tensor, then result will follow.

Now by (1)

$$C'^{\mu}_{\nu\sigma} = A'^{\mu}_{\nu} B'_{\sigma} \quad \text{by (1)}$$

$$= A^{p}_{q} \frac{\partial x'^{\mu}}{\partial x^{p}} \frac{\partial x^{q}}{\partial x'^{\nu}} B_{r} \frac{\partial x^{r}}{\partial x'^{\sigma}}$$

$$= A^{p}_{q} B_{r} \frac{\partial x'^{\mu}}{\partial x^{p}} \frac{\partial x^{q}}{\partial x'^{\nu}} \frac{\partial x^{r}}{\partial x'^{\sigma}}$$

$$= C^{p}_{qr} \frac{\partial x'^{\mu}}{\partial x^{p}} \frac{\partial x^{q}}{\partial x'^{\nu}} \frac{\partial x^{r}}{\partial x'^{\sigma}} \quad \text{by (1)}$$

From it follows that $C^{\mu}_{\nu\sigma}$ is a tensor.

Theorem (8): Show that the open product of two vectors is a tensor of rank 2.

Proof: Let A^{i} be a contravariant vector and B_{j} is a covariant vector.

Also let $C^{i}_{j} = A^{i} B_{j}$ ——— (7)

Then C^{i}_{j} is called open product of A^{i} and B_{j}.

Now by (1) $C'^{i}_{j} = A'^{i} B'_{j} = A^{\alpha} \frac{\partial x'^{i}}{\partial x^{\alpha}} \cdot B_{\beta} \frac{\partial x^{\beta}}{\partial x'^{j}}$

$$C'^{i}_{j} = A^{\alpha} B_{\beta} \frac{\partial x'^{i}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{j}} = C^{\alpha}_{\beta} \frac{\partial x'^{i}}{\partial x^{\alpha}} \cdot \frac{\partial x^{\beta}}{\partial x'^{j}} \quad \text{by (1)}$$

Hence C^{i}_{j} is a tensor of rank 2.

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