Neighbourhood (We will write nbd in short)

A subset U of a metric space X is said to be a neighbourhood of a point $x \in X$ if $\exists$ a real number $r > 0$ such that $S_r(x) \subset U$

Note: (i) Metric space X is nbd of all of its points
(ii) Every open set is nbd of all of its points.
(iii) Since open sphere $S_r(x)$ is open set it is nbd of x.

Hausdorff Property of Metric space

Thm Let (X,d) be any metric space. Then for every pair x & y of distinct points of E, there exists a nbd U of X and a nbd V of y such that $U \cap V = \phi$.

Proof We have $x \neq y$

$\Rightarrow d(x,y) > 0$

Let $r = \frac{d(x,y)}{3}$ i.e, $d(x,y) = 3r$ —— (1)

Let $U = S_r(x)$ & $V = S_r(y)$

Clearly U and V are nbds of x and y respectively, so it is sufficient to show that $U \cap V = \phi$.

Let us assume to the contradiction that $U \cap V \neq \phi$

Let $z \in U \cap V$

$\Rightarrow z \in U$ & $z \in V$

$\Rightarrow z \in S_r(x)$ & $z \in S_r(y)$

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$\Rightarrow d(z,x) < r$ and $d(z,y) < r$ —— (2)

Now $d(x,y) \leq d(x,z) + d(z,y)$

$= d(z,x) + d(z,y)$

$< r + r = 2r$ (From (2) & (1))

Which is a contradiction as $d(x,y) = 3r$ (From (1))

Hence $U \cap V = \phi$ (Proved).

Interior point

Let A be any subset of a metric space (X,d). A point $x \in A$ is said to be interior point of A if exists a real number $r > 0$ such that $S_r(x) \subseteq A$.

Set of all interior points of A is called interior of A. It is denoted by $Int(A)$ or A°

Thm If A is any subset of a metric space (X,d) then prove that $Int(A)$ is an open set.

Proof Let $x \in Int(A)$

$\Rightarrow x$ is interior point of A

$\Rightarrow \exists\ r > 0$ such that $S_r(x) \subseteq A$.

Consider open sphere $S_{r/2}(x)$

Let $y \in S_{r/2}(x)$

$\Rightarrow d(x,y) < r/2$ —— (1)

Now if $z \in S_{r/2}(y)$

$\Rightarrow d(z,y) < r/2$ —— (2)

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$$\Rightarrow d(z,x) \leq d(z,y) + d(y,x)$$
$$< r/_2 + r/_2 = r \quad (\text{from } ① \,\&\, ②)$$

$$\Rightarrow z \in S_r(x)$$
$$\Rightarrow z \in A \quad (\because S_r(x) \subset A)$$

ie $S_{r/2}(y) \subseteq A$

$\Rightarrow y$ is interior point of A

$\Rightarrow y \in Int(A)$

$\Rightarrow S_{r/2}(x) \subseteq Int(A)$

Hence $Int(A)$ is open. (Proved)

Thm A subset A of a metric space X is open if and only if $A = Int(A)$

Proof Let $A = Int(A)$

$\therefore$ By previous theorem, $Int(A)$ is open so A is open.

Conversely: Let A is open set

By definition of interior point, it is clear that $Int(A) \subseteq A$ —— ①

So it is sufficient to prove that
$$A \subseteq Int(A)$$

Let $x \in A$. Since A is open, $\exists\, r > 0$ such that $S_r(x) \subseteq A$.

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$\Rightarrow$ x is interior point of A

$\Rightarrow$ $x \in In(A)$

$\Rightarrow$ $A \subseteq Int(A)$ —— ②

From ① & ② $A = Int(A)$

(Proved)

Th^m Interior of a subset A of metric space A is union of all open subsets of A.

Proof Since Interior of any set A is an open subset of A $\Rightarrow$ $Int(A)$ is contained in union of all open subsets of A

ie $Int(A) \subseteq$ (Union of all open subsets of A) —— ①

Now

Let $x \in$ (Union of all open subsets of A)

$\Rightarrow$ $x \in G$ where G is open subset of A

$\Rightarrow \exists\ r > 0$ such that $S_r(x) \subseteq G$

Also $G \subseteq A$

so $S_r(x) \subseteq A$

$\Rightarrow$ x is interior point of A

$\Rightarrow$ Union of all open subsets of A is subset of $Int(A)$

ie (Union of all open subsets of A) $\subseteq Int(A)$ —— ②

From ① & ② Union of all open subsets of A = $Int(A)$

(Proved)