Tensor Calculus (8)

Contraction: In tensor, if we set one covariant and one contravariant suffixes equal, then that process is called contraction.

Let A^{pq}_{rst} be a tensor of rank five. Then by tensor law of transformation

$$A'^{pq}_{rst} = A^{\alpha\beta}_{ijk} \frac{\partial x'^p}{\partial x^\alpha} \frac{\partial x'^q}{\partial x^\beta} \frac{\partial x^i}{\partial x'^r} \frac{\partial x^j}{\partial x'^s} \frac{\partial x^k}{\partial x'^t}$$

Taking $p = t$, we have

$$A'^{pq}_{rsp} = A^{\alpha\beta}_{ijk} \frac{\partial x'^p}{\partial x^\alpha} \frac{\partial x'^q}{\partial x^\beta} \frac{\partial x^i}{\partial x'^r} \frac{\partial x^j}{\partial x'^s} \frac{\partial x^k}{\partial x'^p}$$

$$= A^{\alpha\beta}_{ijk} \frac{\partial x^k}{\partial x^\alpha} \frac{\partial x'^q}{\partial x^\beta} \frac{\partial x^i}{\partial x'^r} \frac{\partial x^j}{\partial x'^s}$$

But $A^{\alpha\beta}_{ijk} \frac{\partial x^k}{\partial x^\alpha} = A^{\alpha\beta}_{ijk}\, \delta^k_\alpha = A^{\alpha\beta}_{ijk}$

Hence $A'^{pq}_{rsp} = A^{\alpha\beta}_{ijk} \frac{\partial x'^q}{\partial x^\beta} \frac{\partial x^i}{\partial x'^r} \frac{\partial x^j}{\partial x'^s}$, In RHS contains three partial derivatives, so it is a tensor of rank three. Hence A^{pq}_{rsp} is a tensor of rank 3 and type (1,2), while A^{pq}_{rst} is a tensor of rank 5 and type (2,3).

It means that: * Contraction reduces rank of tensor by two.*

Theorem (9): Quotient law of tensors:

Statement: A set of quantities, whose inner product with an arbitrary vector is a tensor, is itself a tensor.

Proof: Let $A^{i_1 i_2 \cdots i_l}_{j_1 j_2 \cdots j_m k}$ be a set of quantities whose inner product with an arbitrary vector u^k

is a tensor of the type $B^{i_1 i_2 \cdots i_l}_{j_1 j_2 \cdots j_m}$.

We have to prove that $A^{i_1 i_2 \cdots i_l}_{j_1 j_2 \cdots j_m k}$ is a tensor.

By assumption $B^{i_1 i_2 \cdots i_l}_{j_1 j_2 \cdots j_m k} = A^{i_1 i_2 \cdots i_l}_{j_1 j_2 \cdots j_m k} u^k$

from which we get

$$B^{\alpha_1 \alpha_2 \cdots \alpha_l}_{\beta_1 \beta_2 \cdots \beta_m} = A^{\alpha_1 \alpha_2 \cdots \alpha_l}_{\beta_1 \beta_2 \cdots \beta_m a} u^a \quad \text{--- (1)}$$

and $$B'^{i_1 i_2 \cdots i_l}_{j_1 j_2 \cdots j_m} = A'^{i_1 i_2 \cdots i_l}_{j_1 j_2 \cdots j_m k} u'^k$$

Since $B^{i_1 i_2 \cdots i_l}_{j_1 j_2 \cdots j_m}$ is a tensor

$$\therefore B^{\alpha_1 \alpha_2 \cdots \alpha_l}_{\beta_1 \beta_2 \cdots \beta_m} \frac{\partial x'^{i_1}}{\partial x^{\alpha_1}} \frac{\partial x'^{i_2}}{\partial x^{\alpha_2}} \cdots \frac{\partial x'^{i_l}}{\partial x^{\alpha_l}} \frac{\partial x^{\beta_1}}{\partial x'^{j_1}} \frac{\partial x^{\beta_2}}{\partial x'^{j_2}} \cdots \frac{\partial x^{\beta_m}}{\partial x'^{j_m}}$$

$$= A'^{i_1 i_2 \cdots i_l}_{j_1 j_2 \cdots j_m k} u'^k.$$

$$\Rightarrow A^{\alpha_1 \alpha_2 \cdots \alpha_l}_{\beta_1 \beta_2 \cdots \beta_m a} u^a \frac{\partial x'^{i_1}}{\partial x^{\alpha_1}} \frac{\partial x'^{i_2}}{\partial x^{\alpha_2}} \cdots \frac{\partial x'^{i_l}}{\partial x^{\alpha_l}} \frac{\partial x^{\beta_1}}{\partial x'^{j_1}} \frac{\partial x^{\beta_2}}{\partial x'^{j_2}} \cdots \frac{\partial x^{\beta_m}}{\partial x'^{j_m}}$$

$$- A'^{i_1 i_2 \cdots i_l}_{j_1 j_2 \cdots j_m k} u'^k = 0$$

$$\Rightarrow A^{\alpha_1 \alpha_2 \cdots \alpha_l}_{\beta_1 \beta_2 \cdots \beta_m a} u'^k \frac{\partial x^a}{\partial x'^k} \frac{\partial x'^{i_1}}{\partial x^{\alpha_1}} \frac{\partial x'^{i_2}}{\partial x^{\alpha_2}} \cdots \frac{\partial x'^{i_l}}{\partial x^{\alpha_l}} \frac{\partial x^{\beta_1}}{\partial x'^{j_1}} \frac{\partial x^{\beta_2}}{\partial x'^{j_2}} \cdots \frac{\partial x^{\beta_m}}{\partial x'^{j_m}}$$

$$- A'^{i_1 i_2 \cdots i_l}_{j_1 j_2 \cdots j_m k} u'^k = 0 \quad (\because u^a \text{ is a vector})$$

$$\Rightarrow u'^k \left[A^{\alpha_1 \alpha_2 \cdots \alpha_l}_{\beta_1 \beta_2 \cdots \beta_m a} \frac{\partial x^a}{\partial x'^k} \frac{\partial x'^{i_1}}{\partial x^{\alpha_1}} \frac{\partial x'^{i_2}}{\partial x^{\alpha_2}} \cdots \frac{\partial x'^{i_l}}{\partial x^{\alpha_l}} \frac{\partial x^{\beta_1}}{\partial x'^{j_1}} \frac{\partial x^{\beta_2}}{\partial x'^{j_2}} \cdots \frac{\partial x^{\beta_m}}{\partial x'^{j_m}} \right.$$

$$\left. - A'^{i_1 i_2 \cdots i_l}_{j_1 j_2 \cdots j_m k} \right] = 0$$

$\because u'^k$ is an arbitrary vector and hence $u'^k \neq 0$

$$\therefore A'^{i_1 i_2 \cdots i_l}_{j_1 j_2 \cdots j_m k} = A^{\alpha_1 \alpha_2 \cdots \alpha_l}_{\beta_1 \beta_2 \cdots \beta_m a} \frac{\partial x^a}{\partial x'^k} \frac{\partial x'^{i_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{i_l}}{\partial x^{\alpha_l}} \frac{\partial x^{\beta_1}}{\partial x'^{j_1}} \cdots \frac{\partial x^{\beta_m}}{\partial x'^{j_m}}$$

Thus $A'^{i_1 i_2 \cdots i_l}_{j_1 j_2 \cdots j_m k}$ is a tensor.