Laplace Transforms of the derivatives of $F(t)$

Theorem: Let $F(t)$ be continuous for all $t \geq 0$ and be of exponential order and if $F'(t)$ is of class A, then Laplace Transform of the derivative $F'(t)$ exists and

$$L\{F'(t)\} = pL\{F(t)\} - F(0)$$

(or)

State and prove Laplace transform of the derivatives of $F(t)$.

Ans:

Statement: Let $F(t)$ be continuous $\forall\, t \geq 0$ and be of exponential order and if $F'(t)$ is of class A, then Laplace transform of the derivative $F'(t)$ exists and

$$L\{F'(t)\} = pL\{F(t)\} - F(0)$$

Proof:

If $F'(t)$ is continuous for all $t \geq 0$ then

$$L\{F'(t)\} = \int_0^\infty e^{-pt} F'(t)\,dt \quad \text{——— (1)}$$

$$= \left[e^{-pt} \cdot F(t)\right]_0^\infty - \int_0^\infty -p e^{-pt} F(t)\,dt$$

[Int. by par

$$= \lim_{t\to\infty} e^{-pt} \cdot F(t) - F(0) + p\int_0^\infty e^{-pt} F(t)\,dt \quad \text{——— (2)}$$

$\because \lim_{t\to\infty} e^{-pt} F(t) = 0$ $\{\because F(t)$ is exponential order$\}$

$\therefore$ (2) becomes

$$L\{F'(t)\} = p\left[L\{F(t)\}\right] - F(0)$$

$$= pL\{F(t)\} - F(0)$$

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Laplace Transform of the nth order derivatives of F(t)

Theorem: Let $F(t)$ and its derivatives $F'(t), F''(t), \ldots, F^{n-1}(t)$ be continuous function for all $t \geq 0$ and be of exponential orders and if $F^n(t)$ is of class A, then Laplace transform of $F^n(t)$ exists and is given by.

$$L\{F^n(t)\} = p^n L\{F(t)\} - p^{n-1}F(0) - p^{n-2}F'(0) - \cdots - F^{n-1}(0)$$

Or

State and prove Laplace transform of nth order derivatives.

Sm.

Statement: Let $F(t)$ and its derivatives $F'(t), F''(t), \ldots, F^{n-1}(t)$ be continuous function for all $t \geq 0$ and be of exponential orders and if $F^n(t)$ is of class A, then Laplace transform of $F^n(t)$ exists and is given by.

$$L\{F^n(t)\} = p^n L\{F(t)\} - p^{n-1}F(0) - p^{n-2}F'(0) - \cdots - F^{n-1}(0)$$

Proof:-

If $F'(t)$ is continuous for all $t \geq 0$ then

$$L\{F'(t)\} = \int_0^\infty e^{-pt} F'(t)\,dt \quad \text{—— (1)}$$

$$= \left[e^{-pt} \cdot F(t)\right]_0^\infty - \int_0^\infty -p e^{-pt} F(t)\,dt \quad \text{[Int. by-parts]}$$

$$= \lim_{t \to \infty} e^{-pt} F(t) - F(0) + p \int_0^\infty e^{-pt} F(t) \quad \text{—— (2)}$$

$\therefore \lim_{t \to \infty} e^{-pt} F(t) = 0$ $\{\because F(t)$ is an exponential order$\}$

$\therefore$ (2) becomes

$$L\{F'(t)\} = 0 - F(0) + p\int_0^{\infty} e^{-pt} F(t)\,dt$$

$$= pL\{F(t)\} - F(0) \quad \text{——— (3)}$$

Applying the result (3) to the 2nd order derivatives $F''(t)$ we have,

$$L\{F''(t)\} = pL\{F'(t)\} - F'(0)$$

$$= p[pL\{F(t)\} - F(0)] - F'(0)$$

$$= p^2 L\{F(t)\} - pF(0) - F'(0) \quad \text{——— (4)}$$

Again applying (3) to the 3rd order derivatives $F'''(t)$ we have,

$$L\{F'''(t)\} = pL\{F''(t)\} - F''(0)$$

$$= p[p^2 L\{F(t)\} - pF(0) - F'(0)] - F''(0)$$

$$= p^3 L\{F(t)\} - p^2 F(0) - pF'(0) - F''(0)$$

Proceeding in this way we get.

$$L\{F^n(t)\} = p^n L\{F(t)\} - p^{n-1}F(0) - p^{n-2}F'(0) - \cdots - F^{n-1}(0)$$

Or

$$L\{F^n(t)\} = p^n L\{F(t)\} - \sum_{r=0}^{n-1} p^{n-1-r} F^r(0)$$

proved

Initial-value theorem

Let $F'(t)$ be continuous $\forall t \geq 0$ and be of exponential order and if $F'(t)$ is of class A then

$$\lim_{t \to 0} F(t) = \lim_{p \to \infty} pL\{F(t)\}$$

Ans

By the Laplace transform of the derivatives of $F(t)$, we have

$$L\{F'(t)\} = \int_0^\infty e^{-pt} F'(t)\,dt$$

$$= pL\{F(t)\} - F(0) \quad \text{--- (1)}$$

Since $F'(t)$ is continuous and of exponential order

$$\therefore \lim_{p \to \infty} L\{F'(t)\} = \lim_{p \to \infty} \int_0^\infty e^{-pt} F'(t)\,dt = 0$$

Taking limit as $p \to \infty$ in (1) we have

$$0 = \lim_{p \to \infty} pL\{F(t)\} - F(0)$$

or, $$F(0) = \lim_{p \to \infty} pL\{F(t)\}$$

$$\therefore \lim_{t \to 0} F(t) = \lim_{p \to \infty} pL\{F(t)\}$$

proved

—x—