

Final - value theorem:

Let $F(t)$ be continuous $\forall t \geq 0$ and be of exponential order and if $F'(t)$ is of class A, then

$$\lim_{t \rightarrow \infty} F(t) = \lim_{p \rightarrow 0} p L\{F(t)\}$$

By Laplace transform of the derivatives of $F(t)$ we have,

$$L\{F'(t)\} = \int_0^{\infty} e^{-pt} F'(t) dt$$

$$= p L\{F(t)\} - F(0) \quad \text{--- (1)}$$

Taking limit as $p \rightarrow 0$ in (1) we have

$$\therefore \lim_{p \rightarrow 0} \int_0^{\infty} e^{-pt} F'(t) dt = \lim_{p \rightarrow 0} p L\{F(t)\} - F(0)$$

$$\Rightarrow \int_0^{\infty} F'(t) dt = \lim_{p \rightarrow 0} p L\{F(t)\} - F(0)$$

$$\Rightarrow [F(t)]_0^{\infty} = \lim_{p \rightarrow 0} p L\{F(t)\} - F(0)$$

$$\Rightarrow \lim_{t \rightarrow \infty} F(t) - \cancel{F(0)} = \lim_{p \rightarrow 0} p L\{F(t)\} - \cancel{F(0)}$$

$$\Rightarrow \lim_{t \rightarrow \infty} F(t) = \lim_{p \rightarrow 0} p L\{F(t)\}$$

—x—

Laplace transform of Integrals

Theorem: If $F(t)$ is piecewise continuous and satisfies

$$|F(t)| \leq M e^{at} \quad \forall t \geq 0$$

for some constant a & M then

Remember $\boxed{L\left\{\int_0^t F(x) dx\right\} = \frac{1}{p} L\{F(t)\}}^*$ ($p > 0, p > a$)

—x—

Multiplication by powers of 't'

Multiplication by t:-

Theorem: If $F(t)$ is a function of class A and if $L\{F(t)\} = f(p)$ Then

$$L\{t F(t)\} = -f'(p)$$

We have,

$$f(p) = L\{F(t)\} = \int_0^{\infty} e^{-pt} F(t) dt$$

$$\therefore f'(p) = \frac{d}{dp} \int_0^{\infty} e^{-pt} F(t) dt$$

$$= \int_0^{\infty} \frac{\partial}{\partial p} \{e^{-pt} F(t)\} dt$$

[By Leibnitz's integral]

$$= - \int_0^{\infty} t e^{-pt} F(t) dt$$

$$= - \int_0^{\infty} e^{-pt} \{t \cdot F(t)\} dt$$

$$\therefore f'(p) = -L\{t \cdot F(t)\}$$

Thus $L\{t \cdot F(t)\} = -f'(p)$ proved.

-x-

(ii) Multiplication by t^n :

Theorem : If $F(t)$ is a function of class A and
if $L\{F(t)\} = f(p)$ then
 $L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n} f(p)$ where $n=1, 2, 3$

Ans : We shall prove this theorem by mathematical induction rule.

When $n=1$

We have,

$$f(p) = L\{F(t)\} = \int_0^{\infty} e^{-pt} F(t) dt$$

$$\therefore f'(p) = \frac{d}{dp} \int_0^{\infty} e^{-pt} F(t) dt$$

$$= \int_0^{\infty} \frac{\partial}{\partial p} \{e^{-pt} F(t)\} dt$$

$$= \int_0^{\infty} -t e^{-pt} F(t) dt$$

$$= - \int_0^{\infty} e^{-pt} \{t \cdot F(t)\} dt$$

$$\text{Thus } f'(p) = -L\{t \cdot F(t)\}$$

$$\therefore L\{t \cdot F(t)\} = -f'(p)$$

$$\Rightarrow L\{t \cdot F(t)\} = (-1) \frac{d}{dp} f(p)$$

i.e.; the theorem is true for $n=1$.

Let us assume that the theorem is true for particular value of n , say m then

$$L\{t^m F(t)\} = (-1)^m \frac{d^m}{dp^m} f(p)$$

$$\Rightarrow \int_0^\infty e^{-pt} \{t^m F(t)\} dt = (-1)^m \frac{d^m}{dp^m} f(p)$$

Now diff both sides wrt p

$$\frac{d}{dp} \int_0^\infty e^{-pt} \{t^m F(t)\} dt = \frac{d}{dp} \left\{ (-1)^m \frac{d^m}{dp^m} f(p) \right\}$$

$$\Rightarrow \int_0^\infty \frac{\partial}{\partial p} \{ e^{-pt} t^m F(t) \} dt = (-1)^m \frac{d^{m+1}}{dp^{m+1}} f(p)$$

$$\Rightarrow - \int_0^\infty t e^{-pt} t^m F(t) dt = (-1)^m \frac{d^{m+1}}{dp^{m+1}} f(p) \quad [\text{By Leibnitz's integral}]$$

$$\Rightarrow \int_0^\infty e^{-pt} t^{m+1} F(t) dt = (-1)^{m+1} \frac{d^{m+1}}{dp^{m+1}} f(p)$$

$$\Rightarrow L\{t^{m+1} F(t)\} = (-1)^{m+1} \frac{d^{m+1}}{dp^{m+1}} f(p)$$

Since the theorem is true for $n=1$ and it is also true for any particular value of $n=m$ and also true for $n=m+1$. Hence it is true for $n=1+1=2$, $n=2+1=3$ etc.

Hence by principle of mathematical induction the theorem is true for every true integral of n .