## Problems based on theorems

(1) Find $L\{t\cos at\}$

Ans:-

$L\{\cos at\} = \frac{p}{p^2+a^2} = f(p)$ (say) if $p>0$

$\therefore L\{t\cos at\} = -f'(p)$

$= -\frac{d}{dp}\left\{\frac{p}{p^2+a^2}\right\}$

$= -\frac{(p^2+a^2) + p2p}{(p^2+a^2)^2} = \frac{-p^2-a^2+2p^2}{(p^2+a^2)^2}$

$= \frac{p^2-a^2}{(p^2+a^2)^2}$

—x—

(2) Find $L\{t^2\sin at\}$

Ans:

$\therefore L\{\sin at\} = \frac{a}{p^2+a^2}$, $p>0$

$= f(p)$ (say)

$\therefore L\{t^2\sin at\} = (-1)^2\frac{d^2}{dp^2}\left(\frac{a}{p^2+a^2}\right)$

$= \frac{d^2}{dp^2}\left(\frac{a}{p^2+a^2}\right) = \frac{d}{dp}\left\{\frac{-2pa}{(p^2+a^2)^2}\right\}$

$= -2a\frac{d}{dp}\frac{p}{(p^2+a^2)^2}$

$= -2a\left[\frac{(p^2+a^2)^2\cdot 1 - p\,2(p^2+a^2)\,2p}{(p^2+a^2)^4}\right]$

$= -2a\left[\frac{p^4+a^4+2p^2a^2-4p^4-4p^2a^2}{(p^2+a^2)^4}\right]$

$\therefore L\{t^2 \sin at\} = \frac{-2a}{(p^2+a^2)^4}\left[-2p^2a^2-3p^4+a^4\right], p>0$

—x—

5) Find $L\{t^2 e^{2t}\}$

$L\{e^{2t}\} = \frac{1}{p-2} \quad , p>2$

$\therefore L\{t^2 e^{2t}\} = (-1)^2 \frac{d^2}{dp^2} \frac{1}{(p-2)}$

$= \frac{d}{dp}\left\{\frac{-1}{(p-2)^2}\right\} = -\left\{\frac{-2}{(p-2)^3}\right\}$

$= \frac{2}{(p-3)^3}$

—x—

Show that $L\{t^3 \cos t\}$

$\because L\{\cos t\} = \frac{p}{p^2+1} \quad , p>0$

$\therefore L\{t^3 \cos t\} = (-1)^3 \frac{d^3}{dp^3}\left\{\frac{p}{p^2+1}\right\}$

$= (-1)\frac{d^2}{dp^2}\left\{\frac{(p^2+1) - p\,2p}{(p^2+1)^2}\right\}$

$= -\frac{d^2}{dp^2}\left\{\frac{1-p^2}{(p^2+1)^2}\right\}$

$= -\frac{d}{dp}\left\{\frac{(p^2+1)^2(-2p) - (1-p^2)2(p^2+1)2p}{(p^2+1)^4}\right\}$

$$\therefore L\{t^3\cos t\} = -\frac{d}{dp}\left[\frac{(p^4+2p^2+1)(-2p) - 4p(1-p^4)}{(p^2+1)^4}\right]$$

$$= -\frac{d}{dp}\left[\frac{-2p^5-4p^3-2p-4p+4p^5}{(p^2+1)^4}\right]$$

$$= -\frac{d}{dp}\left[\frac{2p^5-4p^3-6p}{(p^2+1)^4}\right]$$

$$= -\left[\frac{(p^2+1)^4(10p^4-12p^2-6)-(2p^5-4p^3-6p)4(p^2+1)^3\times 2p}{(p^2+1)^8}\right]$$

$$= -\frac{(p^2+1)^3}{(p^2+1)^8}\left[\frac{(p^2+1)(10p^4-12p^2-6)-8p(2p^5-4p^3-6p)}{1}\right]$$

$$= \frac{-1}{(p^2+1)^5}\left[10p^6-12p^4-6p^2+10p^4-12p^2-6-16p^6+32p^4+48p^2\right]$$

$$= \frac{-1}{(p^2+1)^5}\left[-6p^6+30p^4+30p^2-6\right] \quad \text{Ans.}$$

—x—

5) Evaluate : $L\{e^{-t}t^2\sin 2t\}$

Ans:-

$$\because L\{\sin 2t\} = \frac{2}{p^2+4} = f(p) \quad \text{(say)}$$

$$\therefore L\{e^{-t}\sin 2t\} = f(p+1)$$

$$= \frac{2}{(p+1)^2+4} = \frac{2}{p^2+2p+5}$$

$$\therefore L\{e^{-t}t^2\sin 2t\} = (-1)^2\frac{d^2}{dp^2}\left(\frac{2}{p^2+2p+5}\right)$$

$$\therefore L\{e^{-t}t^2\sin 2t\} = 2\frac{d}{dp}\left\{-\frac{1}{(p^2+2p+5)^2}\times(2p+2)\right\}$$

$$= -2\frac{d}{dp}\left\{\frac{2p+2}{(p^2+2p+5)^2}\right\}$$

$$= -2\left\{\frac{(p^2+2p+5)^2\,2-(2p+2)^2\,2(p^2+2p+5)}{(p^2+2p+5)^4}\right\}$$

$$= -4\frac{(p^2+2p+5)}{(p^2+2p+5)^4}\left[p^2+2p+5-(2p+2)^2\right]$$

$$= \frac{-4}{(p^2+2p+5)^3}\left[p^2+2p+5-4p^2-8p-4\right]$$

$$= \frac{-4}{(p^2+2p+5)^3}\left[-3p^2-6p+1\right]$$

Ans

$$= \frac{(12p^2+24p-4)}{(p^2+2p+5)^3}$$

—x—

Find $L\{\sin at - at\cos at\}$

$$L\{\sin at - at\cos at\} = L\{\sin at\} - aL\{t\cos at\}$$

$$= \frac{a}{p^2+a^2} - a(-1)\frac{d}{dp}\left\{\frac{p}{p^2+a^2}\right\}$$

$$= \frac{a}{p^2+a^2} + a\left[\frac{(p^2+a^2)-p\,2p}{(p^2+a^2)^2}\right]$$

$$= \frac{a}{p^2+a^2} + a\,\frac{a^2-p^2}{(p^2+a^2)^2}$$

$$\therefore L\{\sin at - at\cos at\} = \frac{a(p^2+a^2) + a(a^2-p^2)}{(p^2+a^2)^2}$$

$$= \frac{ap^2+a^3+a^3-ap^2}{(p^2+a^2)^2}$$

$$= \frac{2a^3}{(p^2+a^2)^2} \quad \text{Ans}$$

—x—

## Division by t:

**Theorem:** If $L\{F(t)\} = f(p)$ then

$$L\left\{\frac{1}{t}F(t)\right\} = \int_p^\infty f(p)\,dp$$ , provided the integral exists.

**Proof:**

$$\because L\{F(t)\} = f(p) = \int_0^\infty e^{-pt}F(t)\,dt \quad \text{—— (1)}$$

Int. (1) w.r.t. p from $p=p$ to $p=\infty$, we get

$$\int_p^\infty f(p)\,dp = \int_p^\infty \left[\int_0^\infty e^{-pt}F(t)\,dt\right]dp$$

$$\therefore \int_p^\infty f(p)\,dp = \int_0^\infty \left[\int_p^\infty e^{-pt}\,dp\right]F(t)\,dt$$

{$\because$ p and t are independent variable so order of integration can be change}

$$= \int_0^\infty \left[\frac{e^{-pt}}{-t}\right]_p^\infty F(t)\,dt$$

$$= \int_0^\infty \left[0 + \frac{e^{-pt}}{t}\right]F(t)\,dt$$