$$\therefore \int_p^\alpha f(x)\,dx = \int_0^\alpha e^{-pt} \cdot \frac{F(t)}{t}\,dt$$

$$= L\left\{\frac{F(t)}{t}\right\} \text{ proved}$$

—x—

2016 Q If $L\{F(t)\} = f(s)$, then $L\left\{\frac{F(t)}{t}\right\} = \int_s^\alpha f(s)\,ds$, provided the int. exists.

Ans:- $\because f(s) = L\{F(t)\} = \int_0^\alpha e^{-pt} F(t)\,dt$ —— (1)

Int. (1) w.r.to s from $s = s$ to $s = \alpha$, we get

$$\int_s^\alpha f(s)\,ds = \int_s^\alpha \left[\int_0^\alpha e^{-pt} F(t)\,dt\right] ds$$

$$= \int_0^\alpha \left[\int_s^\alpha e^{-st}\,ds\right] F(t)\,dt$$

$$= \int_0^\alpha \left[\frac{e^{-st}}{-t}\right]_s^\alpha F(t)\,dt$$

$$= \int_0^\alpha \left[0 + \frac{e^{-st}}{t}\right] F(t)\,dt$$

$$= \int_0^\alpha e^{-st} \frac{F(t)}{t}\,dt$$

$$= L\left\{\frac{F(t)}{t}\right\}$$

$$\therefore L\left\{\frac{F(t)}{t}\right\} = \int_s^\alpha f(s)\,ds.$$

Problems:

1) P.T. $L\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$

Ans:

$\because L\{\cos at - \cos bt\} = L\{\cos at\} - L\{\cos bt\}$

$$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} = f(s) \text{ (say)}$$

$\therefore L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty f(x)\,dx$

$$= \frac{1}{2}\int_s^\infty \left\{\frac{2x}{x^2+a^2} - \frac{2x}{x^2+b^2}\right\} dx$$

$$= \frac{1}{2}\left[\log(x^2+a^2)\right]_s^\infty - \frac{1}{2}\left[\log(x^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{x^2+a^2}{x^2+b^2}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{1+\frac{a^2}{x^2}}{1+\frac{b^2}{x^2}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log(1) - \log\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right]$$

$$= -\frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}$$

$$= \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2} \quad \text{proved}$$

$\rightarrow\times$

② Prove that $L\left(\frac{\cos 2t - \cos 3t}{t}\right) = \frac{1}{2}\log\frac{s^2+9}{s^2+4}$

Ans:-

Ans: $L\{\cos 2t - \cos 3t\} = L\{\cos 2t\} - L\{\cos 3t\}$

$$= \frac{s}{s^2+4} - \frac{s}{s^2+9} = f(s) \text{ (say)}$$

$$\therefore L\left\{\frac{\cos 2t - \cos 3t}{t}\right\} = \int_s^\infty f(x)\,dx$$

$$= \int_s^\infty \left\{\frac{x}{x^2+4} - \frac{x}{x^2+9}\right\} dx$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{2x}{x^2+4} - \frac{2x}{x^2+9}\right) dx$$

$$= \frac{1}{2}\left[\log(x^2+4) - \log(x^2+9)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{x^2+4}{x^2+9}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{1+\frac{4}{x^2}}{1+\frac{9}{x^2}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log 1 - \log \frac{1+\frac{4}{s^2}}{1+\frac{9}{s^2}}\right]$$

$$= -\frac{1}{2}\log \frac{s^2+4}{s^2+9}$$

$$= \frac{1}{2}\log\left(\frac{s^2+9}{s^2+4}\right) \quad \text{proved}$$

—x—

(3) Prove that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{p}$ and hence find $L\left\{\frac{\sin at}{t}\right\}$. Does the Laplace transform of $\frac{\cos at}{t}$ exist?

Ans $\therefore L\{\sin t\} = \frac{1}{p^2+1} = f(p)$ (say)

$$\therefore L\left\{\frac{\sin t}{t}\right\} = \int_p^\infty f(x)\,dx$$

$$= \int_p^\infty \frac{1}{x^2+1}\,dx$$

$$= \left[\tan^{-1}x\right]_p^\infty = \left[\tan^{-1}\infty - \tan^{-1}p\right]$$

$$= \frac{\pi}{2} - \tan^{-1}p = \cot^{-1}p$$

$$= \tan^{-1}\frac{1}{p}$$

2nd part:

$$\therefore L\left\{\frac{\sin at}{t}\right\} = a\,L\left\{\frac{\sin at}{at}\right\}$$

$$= a\cdot\frac{1}{a}\tan^{-1}\frac{1}{(p/a)}$$

$\left\{\because L\{f(at)\} = \frac{1}{a}f\left(\frac{p}{a}\right)\right.$
By change scale

3rd part:

$$L\{\cos at\} = \frac{p}{p^2+a^2} = f(p) \text{ (say)}$$

$$\therefore L\left\{\frac{\cos at}{t}\right\} = \frac{1}{2}\int_p^\infty \frac{2x}{x^2+a^2}\,dx$$

$$= \frac{1}{2}\left[\log(x^2+a^2)\right]_p^\infty$$

$$= \frac{1}{2}\lim_{p\to\infty}\log(p^2+a^2) - \frac{1}{2}\log(p^2+a^2)$$

$$\to \infty \quad \text{(not exist)}$$

$\therefore$ The Laplace transform of $\frac{\cos at}{t}$ does not exist.

vvvs

(4) Evaluate $L\left(\frac{1-e^t}{t}\right)$

(or) Find the Laplace transform of $\frac{1-e^t}{t}$

Ans:

$L(1-e^t) = L\{1\} - L\{e^t\}$

$= \frac{1}{p} - \frac{1}{p-1} = f(p)$ (say)

$\therefore L\left(\frac{1-e^t}{t}\right) = \int_p^\infty f(x)\,dx$

$= \int_p^\infty \left(\frac{1}{x} - \frac{1}{x-1}\right) dx = \left[\log x - \log(x-1)\right]_p^\infty$

$= \left[\log \frac{x}{x-1}\right]_p^\infty$

$= \left[\log \frac{1}{1-\frac{1}{x}}\right]_p^\infty = \left[\log(1) - \log\left(1-\frac{1}{x}\right)\right]_p^\infty$

$= \left[\log(1) - \log\frac{p}{p-1}\right]$

$= 0 - \log\frac{p}{p-1}$

$= \log\left(\frac{p-1}{p}\right)$ Ans.

vvvi

remember putting

(5) Show that $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{p}}\, e^{-1/4p}$

Ans:-

Let $F(t) = \sin\sqrt{t}$

then $F'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$ and $F(0) = 0$

$$\because \quad L\{F'(t)\} = pL\{F(t)\} - F(0)$$

$$L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = pL\{\sin\sqrt{t}\} - 0$$

$$\Rightarrow \frac{1}{2} L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = pL\{\sin\sqrt{t}\}$$

$$\Rightarrow L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = 2pL\left[\sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \cdots\right]$$

$$= 2pL\left[t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \cdots\right]$$

$$= 2p\left[L\{t^{1/2}\} - \frac{1}{3!}L\{t^{3/2}\} + \frac{1}{5!}L\{t^{5/2}\} - \cdots\right]$$

$$= 2p\left[\frac{\Gamma(1+\frac{1}{2})}{p^{1+\frac{1}{2}}} - \frac{1}{3!}\frac{\Gamma(1+\frac{3}{2})}{p^{1+\frac{3}{2}}} + \frac{1}{5!}\frac{\Gamma(1+\frac{5}{2})}{p^{1+\frac{5}{2}}} - \cdots\right]$$

$$= 2p\left[\frac{\frac{1}{2}\Gamma(\frac{1}{2})}{p^{3/2}} - \frac{1}{3!}\frac{\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}}{p^{5/2}} + \frac{1}{5!}\frac{\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})}{p^{7/2}} - \cdots\right]$$

$$= 2p\left[\frac{\frac{1}{2}\sqrt{\pi}}{p^{3/2}} - \frac{1}{6}\frac{3}{4}\frac{\sqrt{\pi}}{p^{5/2}} + \frac{1}{120}\frac{15}{8}\frac{\sqrt{\pi}}{p^{7/2}} - \cdots\right]$$

$$= \cancel{2}p\frac{1}{\cancel{2}}\frac{\sqrt{\pi}}{p^{3/2}}\left[1 - \frac{1}{4}\frac{1}{p} + \frac{1}{32}\frac{1}{p^2} - \cdots\right]$$

$$= \frac{\sqrt{\pi}}{p^{1/2}}\left[1 - \frac{1}{4p} + \frac{1}{2!}\left(\frac{1}{4p}\right)^2 - \cdots\right]$$

$$\therefore L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{p^{1/2}} e^{-\frac{1}{4p}}$$

proved