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Q: Prove that $o(a) = o(x^{-1}ax)$

Ans: Let a and x be arbitrary elements of a group G and e is its identity elements.

Let $o(a) = n$, $o(x^{-1}ax) = m$

to prove that $o(a) = o(x^{-1}ax)$

i.e $m = n$ —— (1)

$$(x^{-1}ax)^2 = (x^{-1}ax)(x^{-1}ax)$$
$$= x^{-1}a(xx^{-1})ax = x^{-1}aeax$$
$$= x^{-1}aax = x^{-1}a^2x \text{ —— (2)}$$

$$\Rightarrow (x^{-1}ax)^3 = (x^{-1}ax)^2(x^{-1}ax) = (x^{-1}a^2x)(x^{-1}ax)$$
$$= x^{-1}a^2(xx^{-1})ax$$
$$= x^{-1}a^2eax$$
$$= x^{-1}a^2ax = x^{-1}a^3x \text{ —— (3)}$$

Progressing in this way we get:

$$(x^{-1}ax)^n = x^{-1}a^nx \text{ —— (4)}$$

$o(a) = n \Rightarrow a^n = e$

$\Rightarrow (x^{-1}ax)^n = x^{-1}a^nx = x^{-1}ex = x^{-1}x = e$

$\Rightarrow (x^{-1}ax)^n = e$

$\Rightarrow o(x^{-1}ax) \leq n$

$\Rightarrow m \leq n$ —— (5)

$\because o(x^{-1}ax) = m \Rightarrow (x^{-1}ax)^m = e$

$\Rightarrow x^{-1}a^mx = x^{-1}x$ $\{\because x^{-1}x = e\}$

$\Rightarrow a^mx = x$ [by left cancellation law]

$\Rightarrow a^m x = e x$

$\Rightarrow a^m = e$ [by left right concellation law]

$\Rightarrow O(a) \leq m$

$\Rightarrow n \leq m$ —— (6)

Combining (5) and (6), we get

$n = m$

(4) Theorem :- If a and b are arbitarary elements of a group G then order of 'ab' is the same as order of 'ba'

or.

Prove that $O(ab) = O(ba)$

Ans → Let a and b be arbitrary elements of a group G.

to prove $O(ab) = O(ba)$

$ba = (eb)a = (a^{-1}ab)a$ $\{\because e = a^{-1}a\}$

$= a^{-1}(ab)a$

$\Rightarrow O(ba) = O\{a^{-1}(ab)a\}$ —— (1)

we know

$O(a) = O(x^{-1}ax)$

$\therefore O(ab) = O\{x^{-1}(ab)x\}$

$\Rightarrow O(ab) = O\{a^{-1}(ab)a\}$ —(2) {taking $x = a$}

from (1) and (2)

$O(ab) = O(ba)$

(5) The order of a^{-1} is the same as that of $a \in G$

or

$O(a^{-1}) = O(a)$

Ans: Let a be an arbitrary element of a group G.

Let $O(a) = n$, $O(a^{-1}) = m$

To prove that

$m = n$

$o(a) = n \Rightarrow a^{n} = e \Rightarrow (a^{n})^{-1} = e^{-1} = e$

$\Rightarrow a^{-n} = e$

$\Rightarrow (a^{-1})^{n} = e$

$\Rightarrow o(a^{-1}) \leq n \Rightarrow m \leq n$ —— (1)

$o(a^{-1}) = m \Rightarrow (a^{-1})^{m} = e \Rightarrow a^{-m} = e \Rightarrow (a^{m})^{-1} = e$

$\Rightarrow a^{-m} = e \Rightarrow (a^{-m})^{-1} = e^{-1}$

$\Rightarrow a^{m} = e$

$\Rightarrow o(a) \leq m$

$\Rightarrow n \leq m$ —— (2)

combining (1) and (2) we have $n = m$

$m = n$

Proved

(6) ~~Theorem~~ The order of any element of a finite group is fin

Proof

Let a be an arbitrary element of a finite group G.

To prove $o(a)$ is finite.

By closure property all elements $a^2, a^3, \ldots$ belong to G

That is to say $a, a^2, a^3, a^4, a^5 \ldots$ etc. belong to G.

But these elements are not all distinct, Since G is finite.

Let 'e' be the identity in G, then $a^0 = e$

All the elements of G are not distinct implies that we can write.

$a^m = a^n$ where $m > n$ and m, n are finite

$\Rightarrow a^m a^{-n} = a^n a^{-n}$

$\Rightarrow a^{m-n} = a^0 = e$

$\Rightarrow a^p = e$.

$\{ \because m > n, m - n = p \}$

$\Rightarrow p > 0$

Also m and n are finite and hence p is a finite possitive integer.

Now p is a possitive integer.

Such that $a^p = e$

This proves that $o(a) \leq p$ = finite number.

i.e $o(a) \leq$ a finite number or $o(a)$ is finite.

Q If a group contain even number of elements

(or If a group of even order) then prove that there exist least one element a in G such that $a^2 = e$] or $a = a^{-1}$

(or Such that order of a is 2)

Ans: Let G be a group of even order, so G contains e number of elements.

We know each element of a group has unique inver the identity element e is its own inverse. Hence t exist at least one element a (say) in G such th

$a = a^{-1}$

$\Rightarrow aa = aa^{-1}$

$\Rightarrow a^2 = e$

(2) Prove that a group of order 3 is abelian.

Ans Let G be a group of order 3.
Let {e, a, b} be three elements of G,
where e is the identity element of G
To prove G is abelian
The composite table of G as given under :

·	e	a	b
e	e	a	b
a	a	a^2	ab
b	b	ba	b^2

Since each element in any row or column of table occurs once and only once.
Hence from this table we see

$ab = e$ and $ba = e$ $\left\{ \because ab \neq a, b \text{ [illegible]} \right.$ and $ba \neq a, b$

$\Rightarrow ab = ba$

$\Rightarrow$ G is abelian.

(3) Prove that a group having four elements is abelian

Ans Let G be a group containing four elements, say e, a, b, c, where e is the identity element.
To prove G is abelian
The composite table of G as under :-

·	e	a	b	c
e	e	a	b	c
a	a	a^2	ab	ac
b	b	ba	b^2	bc
c	c	ca	cb	c^2

Since each element in any row or column of this table occurs once and only once.

So $ab = c$ or $ab = e$ $\{\because ab \neq a, b\}$
and $ba = c$ or $ba = e$ $\{\because ba \neq a, b\}$

If $ab = e$ $\Rightarrow cb = a$ [From table $cb \neq c, b, e$]
$\Rightarrow b^2 = c$ [" "]
$\Rightarrow ba = e$ [" "]

~~Again if~~ Similarly if $ab = c \Rightarrow ba = c$

$\therefore ab = ba$

Similarly, we can show that $ac = ca$ and $bc = cb$

Hence G is abelian.

(4) Q. Prove that a group of ~~five~~ four or fewer elements is abelian.

Ans: If a group contain only one element then it be identity element e and since $ee = e$

So it must be abelian

We know a group of order prime is cyclic and every cyclic group necessarily abelian.

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So if a group contain 2 or 3 elements, which are p... so G is cyclic and so abelian. It remains to prove that a g... contains namely e, a, b, c, where e is the identity e...

then Do (3)

⑤ P.T. a group of five or fewers elements is abelian.

Ans: Same as four (4) [only write 2,3,5 are prime]

Cyclic group :- If G be a group and a be an elements of G such that every elements of G is some integral power of a, then G is called cyclic group generated by a.
Here a is called generator of group G

Ex. ① $G=\{1,\omega,\omega^2\}$ is a cyclic group generated by ω or ω^2

$\because 1=\omega^3$

and $\omega^2=\omega^2$

(2) $G=\{1,-1,i,-i\}$ is a cyclic group generated by i or $-i$

$\because 1=i^4$

$-1=i^2$, $i^3=-i$

(2m)

① Theorem P.T. every cyclic group is necessarily abelian.

Ans → Let $G=\{a\}$ be a cyclic group generated by a.
Let x and y be two arbitrary elements of G
Since G is cyclic generated by a
So x and y must be some integral power of a
say $x=a^m$, $y=a^n$

$\therefore xy = a^m a^n = a^{m+n}$

$= a^{n+m} = a^n a^m = yx$

Since x,y are taken arbitrary and so we can say G is abelian.

—x—

Note: If G is cyclic generated by a, then in short we write $G=\{a\}$ is cyclic group

—x—