

Tensor Calculus (12)

Definition: Associate tensors: We define $A_\mu = g_{\mu\alpha} A^\alpha$ (1)
Then the tensor A_μ is called associate to A^α . Also we say that the tensors (vectors) A_μ and A^μ are associate to each other.

We also define $A^\mu = g^{\mu\alpha} A_\alpha$ (2) This is called raising the subscript.

Multiplying (2) by $g_{\mu\beta}$, we have

$$g_{\mu\beta} A^\mu = g_{\mu\beta} g^{\mu\alpha} A_\alpha = \delta_\beta^\alpha A_\alpha = A_\beta$$

$$\text{or } A_\beta = g_{\mu\beta} A^\mu = g_{\alpha\beta} A^\alpha = g_{\beta\alpha} A^\alpha$$

$$\text{or } A_\beta = g_{\beta\alpha} A^\alpha \text{ or } A_\mu = g_{\mu\alpha} A^\alpha, \text{ This is eq (1).}$$

This is called lowering subscript.

Thus there are three processes.

(i) Multiplication by $g^{\mu\alpha}$ gives substitution with raising.

(ii) Multiplication by $g_{\mu\alpha}$ gives substitution with lowering.

(iii) Multiplication by $g_{\mu\alpha}$ gives a simple substitution.

Definition: Magnitude of a vector: The magnitude A of a vector A_α is defined as $A^2 = g^{\alpha\beta} A_\alpha A_\beta$.

$$\text{Evidently } A^2 = g^{\alpha\beta} A_\alpha A_\beta = A^\beta A_\beta = A^\beta g_{\alpha\beta} A^\alpha = g_{\alpha\beta} A^\alpha A^\beta$$

$$\text{or } A^2 = g_{\alpha\beta} A^\alpha A^\beta. \text{ This shows that magnitude}$$

of contravariant component and covariant component of the same vector are equal.

Angle between two vectors: Let θ is the angle between

any two vectors A^α and B^α .

Then we define:

$$\cos \theta = \frac{g_{\alpha\beta} A^\alpha B^\beta}{\sqrt{\{ (g_{\alpha\beta} A^\alpha A^\beta) \} \cdot \{ (g_{\mu\nu} B^\mu B^\nu) \}}}$$

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Question ① Show that Kronecker delta is a mixed tensor of rank two.

Ans: Let A^α be an arbitrary vector, then its inner product with Kronecker delta is

$$\delta_\alpha^\mu A^\alpha = A^\mu = \text{tensor of rank one.}$$

Hence by quotient law of tensors, δ_α^μ is a tensor. Since it is covariant of order one and contravariant of order one. Hence it is a mixed tensor of rank two.

Question ② Prove that (i) $dg_{\alpha\beta} = -g_{\mu\alpha} g_{\nu\beta} dg^{\mu\nu}$.

(ii) $A^\alpha B^\beta dg_{\alpha\beta} = -A_{\alpha\beta} A^{\alpha\beta}$ (iii) $\frac{dg}{g} = -g_{\mu\nu} dg^{\mu\nu}$.

Ans (i) Since $g_{\mu\alpha} g^{\mu\nu} = \delta_\alpha^\nu = 1$ or 0

Taking differential of both sides

$$dg_{\mu\alpha} g^{\mu\nu} + g_{\mu\alpha} dg^{\mu\nu} = 0$$

Multiplying both sides by $g_{\nu\beta}$.

$$g_{\nu\beta} g^{\mu\nu} dg_{\mu\alpha} = -g_{\nu\beta} g_{\mu\alpha} dg^{\mu\nu}$$

$$\Rightarrow \delta_\beta^\mu dg_{\mu\alpha} = -g_{\mu\alpha} g_{\nu\beta} dg^{\mu\nu}$$

$$\Rightarrow dg_{\beta\alpha} = -g_{\mu\alpha} g_{\nu\beta} dg^{\mu\nu}$$

$$\Rightarrow dg_{\alpha\beta} = -g_{\mu\alpha} g_{\nu\beta} dg^{\mu\nu}, \text{ for } g_{\alpha\beta} \text{ is symmetric.}$$

Ans (ii) First, Prove the result (i), then multiplying the result (i) by $A^\alpha B^\beta$. Proved.

$$\begin{aligned} A^\alpha B^\beta dg_{\alpha\beta} &= -A^\alpha B^\beta g_{\mu\alpha} g_{\nu\beta} dg^{\mu\nu} = -A^\alpha B^\beta g_{\nu\beta} g_{\mu\alpha} dg^{\mu\nu} \\ &= -A^\alpha g_{\mu\alpha} g_{\nu\beta} dg^{\mu\nu} = -A^\alpha g_{\mu\alpha} dg^{\mu\nu} \\ &= -A_{\mu\alpha} dg^{\mu\alpha} = -A_{\mu\nu} dg^{\mu\nu}, \text{ for } A_{\mu\nu} \text{ is symmetric.} \\ &= -A_{\alpha\beta} dg^{\alpha\beta}, \text{ Proved.} \end{aligned}$$