General Method of Solution of PDE of first order

When a given differential equation can not be reduced to any of four forms discussed in previous classes, we use following method:

Charpits Method:

Let the given differential equation be

$$f(x, y, z, p, q) = 0 \quad \text{——— (1)}$$

Let us find another relation

$$F(x, y, z, p, q) = 0 \quad \text{——— (2)}$$

On Solving (1) & (2) and finding p & q, we substitute it in the equation

$$z = p\,dx + q\,dy \quad \text{——— (3)}$$

Differentiation (1) & (2) partially w.r.to x & y and eliminating $\frac{\partial p}{\partial x}$, $\frac{\partial q}{\partial y}$ and $\frac{\partial p}{\partial y}\left(=\frac{\partial q}{\partial x}\right)$ we get

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial q} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z}$$
$$+ \left(-\frac{\partial f}{\partial p}\right)\frac{\partial F}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial y} = 0 \quad \text{— (4)}$$

Which is linear differential equation of first order with x, y, z, p, q as independent variable and F as dependent variable. The auxiliary eqn are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0} \quad \text{— (5)}$$

Take the simplest relation involving atleast one of p & q for $F=0$ & from $F=0$ & $f=0$ find values of p & q and putting in $dz = p\,dx + q\,dy$ which on integrating gives the solution.

Example 1. Solve $(p^2+q^2)y = qz$. ——— ①

Sol^n Here $f \equiv (p^2+q^2)y - qz = 0$.

The charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{\frac{\partial f}{\partial q}} = \frac{dF}{0}.$$

$$\Rightarrow \frac{dp}{-pq} = \frac{dq}{(p^2+q^2)-q^2} = \frac{dz}{-2p^2y - 2q^2y + qz} = \frac{dx}{-2py} = \frac{dy}{-2qy+z} = \frac{dF}{0}$$

Taking the first two members, we have.

$$\frac{dp}{-pq} = \frac{dq}{p^2}$$

$$\Rightarrow p\,dp = -q\,dq$$

On integrating we get

$$\frac{p^2}{2} = -\frac{q^2}{2} + \frac{a^2}{2} \quad \text{where } \frac{a^2}{2} \text{ is const. of integration.}$$

$$\Rightarrow p^2 + q^2 = a^2 \quad \text{——— ②}$$

Putting in ① we get

$$a^2 y = qz$$

$$\Rightarrow q = \frac{a^2 y}{z}$$

Putting value of q in ② we get

$$p^2 + \frac{a^4 y^2}{z^2} = a^2$$

$$\Rightarrow p^2 = a^2\left(1 - \frac{a^2y^2}{z^2}\right)$$

$$\Rightarrow p^2 = \frac{a^2}{z^2}(z^2 - a^2y^2)$$

$$\Rightarrow p = \frac{a}{z}\sqrt{z^2 - a^2y^2} \quad \text{(Taking +ve root)}.$$

From relation

$$dz = p\,dx + q\,dy$$

$$\Rightarrow dz = \frac{a}{z}\sqrt{z^2 - a^2y^2}\,dx + \frac{a^2y}{z}\,dy$$

$$\Rightarrow z\,dz = a\sqrt{z^2 - a^2y^2}\,dx + a^2y\,dy$$

$$\Rightarrow z\,dz - a^2y\,dy = a\sqrt{z^2 - a^2y^2}\,dx$$

$$\Rightarrow \frac{z\,dz - a^2y\,dy}{\sqrt{z^2 - a^2y^2}} = a\,dx$$

$$\Rightarrow \frac{1}{2}\frac{d(z^2 - a^2y^2)}{\sqrt{z^2 - a^2y^2}} = a\,dx$$

On integrating we get

$$\sqrt{z^2 - a^2y^2} = ax + b$$

$$\Rightarrow z^2 - a^2y^2 = (ax+b)^2$$

$$\Rightarrow z^2 = a^2y^2 + (ax+b)^2 \quad \text{——— (3)}$$

Which is the required complete solⁿ

Now to find general solⁿ we write $b = \phi(a)$ in the complete solⁿ $z = a^2y^2 + [ax + \phi(a)]^2$ ——— (4)

Defferentiating (4) partially w.r to a we get

$$0 = 2ay^2 + 2[ax + \phi(a)] \cdot [x + \phi'(a)] \quad \text{--- (5)}$$

The general solⁿ is obtained by eliminating a from (4) and (5)

Now to find singular solution we defferentiate eqn (3) partially with respect to a and w.r to b

we get $0 = 2ay^2 + 2(ax+b) \cdot x$ --- (6)

& $0 = 2(ax+b)$ --- (7)

Now eliminating a & b from (3), (6) & (7)

we get $z = 0$ which clearly satisfies the given eqn (1) and therefore $z = 0$ is singular solution.

Some problem for practice

(1) $px + qy = pq$

(2) Apply charpits method to find complete solution of $z = px + qy + p^2 + q^2$

(3) $2zx - px^2 - 2qxy + pq = 0.$

(4) $z^2(p^2z^2 + q^2) = 1$