Ans (iii) Since we know that $g_{\mu\nu} g^{\mu\nu} = 0$,

Taking differential of both sides

$$g_{\mu\nu} dg^{\mu\nu} + g^{\mu\nu} dg_{\mu\nu} = 0$$

$$\Rightarrow -g_{\mu\nu} dg^{\mu\nu} = g^{\mu\nu} dg_{\mu\nu} \quad \text{—— (1)}$$

We also know that $\frac{\partial a}{\partial x^k} = A^i_j \frac{\partial a^i_j}{\partial x^k}$

In our case this becomes $\frac{\partial g}{\partial x^k} = \text{cofactor of } g_{ij} \frac{\partial g_{ij}}{\partial x^k}$

But $g^{ij} = \frac{\text{Cofactor of } g_{ij}}{g}$, by definition of Reciprocal tensor.

$$\therefore \frac{\partial g}{\partial x^k} = g . g^{ij} \frac{\partial g_{ij}}{\partial x^k} \Rightarrow \frac{\partial g}{\partial x^k} dx^k = g . g^{ij} \frac{\partial g_{ij}}{\partial x^k} dx^k$$

$$\Rightarrow dg = g . g^{ij} dg_{ij} \Rightarrow \frac{dg}{g} = g^{ij} dg_{ij} = g^{\mu\nu} dg_{\mu\nu}. \quad \text{—— (2)}$$

By (1) and (2), $\frac{dg}{g} = -g_{\mu\nu} dg^{\mu\nu}$, Proved.

Question (2) Show that $\sqrt{g}\, dx^1 dx^2 \ldots dx^n$ is an invariant

Ans Since g_{ij} is a second rank covariant tensor

so that $g'_{ij} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^i} \frac{\partial x^\beta}{\partial x'^j}$

Taking determinant of both sides

$$|g'_{ij}| = |g_{\alpha\beta}| \left|\frac{\partial x^\alpha}{\partial x'^i}\right| \left|\frac{\partial x^\beta}{\partial x'^j}\right|.$$

$$\Rightarrow g' = g \left|\frac{\partial x}{\partial x'}\right|^2$$

$$\Rightarrow \sqrt{g'} = \sqrt{g} \left|\frac{\partial x}{\partial x'}\right| \quad \text{—— (1)}$$

But, $\left|\frac{\partial x}{\partial x'}\right| dx'^1 dx'^2 \ldots dx'^n = dx^1 dx^2 \ldots dx^n$ —— (2)

S. Altashui
K.C.C.
29.06.2020

Eliminiting $\left|\frac{\partial x}{\partial x'}\right|$ from ① and ②

$$\frac{\sqrt{g'}}{\sqrt{g}}\, dx'^1 dx'^2 \cdots dx'^n = dx^1 dx^2 \cdots dx^n$$

$$\Rightarrow \sqrt{g'}\, dx'^1 dx'^2 \cdots dx'^n = \sqrt{g}\, dx^1 dx^2 \cdots dx^n$$

This proves that $\sqrt{g}\, dx^1 dx^2 \cdots dx^n$ is invariant.
Proved.

Question ③ If A^i and B^i are contravariant vectors and $C_{ij} A^i B^j$ is an invariant. Prove that C_{ij} is a tensor of the second order.

Ans: Let A^i and B^i are Contravariant vectors. Also suppose that $C_{ij} A^i B^j$ is an invariant so that $C_{ij} A^i B^j = C'_{ij} A'^i B'^j$ ——— ①

to prove that C_{ij} is a tensor.

Then by ①

$$C_{\alpha\beta} A^\alpha B^\beta = C'_{ij} A'^i B'^j = C'_{ij} A^\alpha \frac{\partial x'^i}{\partial x^\alpha} B^\beta \frac{\partial x'^j}{\partial x^\beta}$$

$$\Rightarrow A^\alpha B^\beta \left(C_{\alpha\beta} - C'_{ij} \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x'^j}{\partial x^\beta} \right) = 0$$

$$\Rightarrow C_{\alpha\beta} - C'_{ij} \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x'^j}{\partial x^\beta} = 0 \quad (\because A^\alpha \neq 0, B^\beta \neq 0)$$

$$\Rightarrow C_{\alpha\beta} = C'_{ij} \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x'^j}{\partial x^\beta}$$

$\Rightarrow C_{ij}$ is a second rank covariant tensor.

— x — x —