Euclidean Ring or Euclidean domain

Let R be an integral domain. It is said to be a Euclidean domain if for every non-zero element $a \in R$ we can assign a non-negative integer $d(a)$ such that

(i) for all $a, b \in R$, both non-zero, $d(ab) \geq d(a)$

(ii) for any $a, b \in R$, $b \neq 0$, $\exists$ q & $r \in R$ such that

$$a = qb + r \text{ where either } r = 0 \text{ or } d(r) < d(b)$$

Second part of definition is called division algorithm.

Ex 1. The ring of integer is a Euclidean ring.

Sol^n Let $(I, +, \cdot)$ be the ring of integers where

$$I = \{ \cdots -2, -1, 0, 1, 2, 3, \ldots \}$$

Let d-function on non-zero elements of I be defined as $d(a) = |a|$ $\forall$ $0 \neq a \in I$.

We have for $a, b \in I$

$$|ab| = |a| \cdot |b|$$

$$\Rightarrow |ab| \geq |a| \quad (\because |b| \geq 1 \text{ if } 0 \neq b \in I)$$

$$\Rightarrow d(ab) \geq d(a)$$

Also we know that if $a, b \in I$ where $b \neq 0$ $\exists$ integers q & r such that

$$a = qb + r \quad \text{where} \quad 0 \leq r < |b|$$

ie either $r = 0$ or $0 < r < |b|$

ie either $r = 0$ or $|r| < |b|$ $(\because r = |r|)$

ie either $r=0$ or $d(r)<d(b)$

Hence $(I,+,\cdot)$ is a Euclidean ring.

Ex 2 The ring of polynomials over a field is a Euclidean ring.

Sol Let F be a field and $F[x]$ be ring of polynomial over field F.

Let us define d-function on non-zero polynomial in $F[x]$ as

$$d[f(x)] = \deg f(x) \quad \forall \ 0 \neq f(x) \in F[x]$$

Obviously $d[f(x)]$ is non-negative as degree of a polynomial is always non-negative

Let $f(x), g(x)$ be two non-zero polynomials in $F[x]$ then $\deg[f(x).g(x)] = \deg f(x) + \deg g(x)$

$$\Rightarrow \deg[f(x).g(x)] \geq \deg f(x) \quad [\because \deg g(x) \geq 0]$$

$$\Rightarrow \quad d[f(x).g(x)] \geq d[f(x)]$$

Also for any two polynomials $f(x)$ & $g(x) \in F[x]$ where $g(x) \neq 0$, $\exists$ two polynomials $q(x)$ & $r(x)$ in $F[x]$ such that $f(x) = q(x).g(x) + r(x)$ where either $r(x) = 0$ or $\deg r(x) < \deg g(x)$

ie either $r(x) = 0$ or $d[r(x)] < d[g(x)]$

Hence ring of polynomials over a field is Euclidean domain.

Ex 3. Every field is a Euclidean ring

Soln: Let F be a field. Let us define a d-function on non-zero elements of F as

$$d(a) = 0 \quad \forall \; 0 \neq a \in F.$$

Thus we have assigned a non-zero integer $d(a)$ to every non-zero elements of F.

Let a, b be any two non-zero elements of F

$$\Rightarrow ab \neq 0$$

$$\Rightarrow d(ab) = 0 = d(a)$$

Thus we have $d(ab) \geq d(a)$

Also if $a, b \in F$ where $b \neq 0$ then we can write $a = (ab^{-1})b + 0$

ie $a = qb + r$ where $q = ab^{-1}$ & $r = 0$

Hence F is a Euclidean ring.

Ex. 3 The ring of Gaussian integers is Euclidean ring.

Soln: Let $(G, +, \cdot)$ be ring of Gaussian integers.

where $G = \{x + iy \mid x, y \in I\}$.

Let us define d function as

$$d(x+iy) = x^2 + y^2 \quad \forall \quad 0 \neq x+iy \in G.$$

Obviously d assigns a non-negative integer to every non-zero element of G.

Let $x+iy$ & $m+in$ be two non-zero elements of G

then $d[(x+iy)(m+in)] = d[(xm-yn)+i(my+nx)]$

$$= (xm-yn)^2 + (my+nx)^2$$

$$= x^2m^2 + y^2n^2 - \cancel{2xmyn} + m^2y^2 + n^2x^2 + \cancel{2myxn}$$

$$= x^2(m^2+n^2) + y^2(m^2+n^2)$$

$$= (m^2+n^2)(x^2+y^2)$$

$$\geq (x^2+y^2) = d(x+iy)$$

Hence $d[(x+iy)(m+in)] \geq d(x+iy)$

Let $\alpha, \beta \in G$ where $\beta \neq 0$.

Let $\alpha = x+iy$ and $\beta = m+in$.

Let us define a complex number λ

$$\lambda = \frac{\alpha}{\beta} = \frac{x+iy}{+i}$$

$$\Rightarrow \lambda = \frac{x+iy}{m+in} \times \frac{m-in}{m-in}$$

$$\Rightarrow \lambda = \frac{(mx+ny)+i(my-nx)}{m^2+n^2}$$

$$\Rightarrow \lambda = \frac{mx+ny}{m^2+n^2} + i\,\frac{my-nx}{m^2+n^2}$$

$\Rightarrow \lambda = p+iq$ where p & q are rational nos.

Thus we have

$\alpha = \lambda'\beta + (\lambda - \lambda')\beta$ where λ' and $(\lambda-\lambda')\beta$ are Gaussian integers and either $(\lambda-\lambda')\beta = 0$ or $d[(\lambda-\lambda')\beta] < d(\beta)$.

Hence ring of Gaussian integers is Euclidean ring.