

① GENERALISED CO-ORDINATES:→

A Set of independent variables, sufficient in number to specify completely the configuration of a system of particles is called a set of co-ordinates of the system. If these co-ordinates are expressed in a most general way without referring to any particular co-ordinate system, then these are termed as "Generalised co-ordinates" of the system, which are given as $q_1, q_2, q_3, \dots, q_k, \dots, q_n$. Where "n" denotes the number of degree of freedom of the system.

② GENERALISED FORCE

The generalised force (Q_k) corresponding to the generalised co-ordinate q_k of a system is mathematically defined by,

$$Q_k = - \frac{\partial V}{\partial q_k}$$

Where "V" being the total potential energy of the system.

③ GENERALISED MOMENTUM

The generalised momentum (P_k) of a system corresponding to the generalised co-ordinates q_k of that system is defined by,

$$P_k = \frac{\partial h}{\partial \dot{q}_k}$$

Where "h" is the ~~ham~~ hamiltonian of the system.

LANGRANGES EQUATION

DEFⁿ → The Lagrangian " L " of a system of particles is defined by the difference of the kinetic energy (T) and the potential energy (V) of that system.

$$\text{i.e. } \boxed{L = \{T - V\}}$$

In order to obtain the Lagrangian equation of a system of particles, let us consider k^{th} particle of that system, having position-vector \vec{r}_k , which according to transformation equation may be expressed as,

$$\vec{r}_k = \vec{r}_k(q_1, q_2, q_3, \dots, q_n, t) \quad \text{--- (1)}$$

where $q_1, q_2, q_3, \dots, q_n$ are the generalised co-ordinates of the system.

Differentiating eqn (1) w.r.t. time, we get,

$$\dot{\vec{r}}_k = \sum_{\alpha=1}^n \frac{\partial \vec{r}_k}{\partial q_\alpha} \cdot \dot{q}_\alpha \quad \text{--- (2)}$$

[If the system does not contain time explicitly]

Further differentiating eqn (2) w.r.t. \dot{q}_α we get,

$$\boxed{\frac{\partial \dot{\vec{r}}_k}{\partial \dot{q}_\alpha} = \frac{\partial \vec{r}_k}{\partial q_\alpha}} \quad \text{--- (3)}$$

[As $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n$ are independent of \dot{q}_α]

If k^{th} particle of mass m_k of the system acquires an acceleration $\ddot{\vec{r}}_k$ under the action of the force \vec{F}_k ,

then from D'Alembert's Principle, we get,

$$\sum_k m_k \ddot{\vec{r}}_k \cdot \frac{\partial \vec{r}_k}{\partial q_\alpha} = \sum_k \vec{F}_k \cdot \frac{\partial \vec{r}_k}{\partial q_\alpha} \quad \text{--- (4)}$$

L.H.S. of Equation (4)

$$\begin{aligned}
 \sum_k m_k \vec{r}_k \cdot \frac{\partial \vec{r}_k}{\partial q_\alpha} &= \sum_k m_k \left[\frac{d}{dt} \left(\vec{r}_k \cdot \frac{\partial \vec{r}_k}{\partial \dot{q}_\alpha} \right) - \vec{r}_k \cdot \frac{\partial \vec{r}_k}{\partial q_\alpha} \right] \\
 &= \sum_k m_k \left[\frac{d}{dt} \left(\vec{r}_k \cdot \frac{\partial \vec{r}_k}{\partial \dot{q}_\alpha} \right) - \vec{r}_k \cdot \frac{\partial \vec{r}_k}{\partial q_\alpha} \right] \\
 &\quad \text{By using equation (3)} \\
 &= \sum_k m_k \left[\frac{d}{dt} \left(\frac{1}{2} \frac{\partial \vec{r}_k^2}{\partial \dot{q}_\alpha} \right) - \frac{1}{2} \frac{\partial \vec{r}_k^2}{\partial q_\alpha} \right] \\
 &= \frac{d}{dt} \frac{\partial}{\partial \dot{q}_\alpha} \left(\sum_k \frac{1}{2} m_k \vec{r}_k^2 \right) - \frac{\partial}{\partial q_\alpha} \left(\sum_k \frac{1}{2} m_k \vec{r}_k^2 \right) \\
 &= \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha}
 \end{aligned}$$

As, $\sum_k \frac{1}{2} m_k \vec{r}_k^2 = T$, The total kinetic energy of the system.

Where as; R.H.S. of equation (4)

$$\sum_k \vec{F}_k \cdot \frac{\partial \vec{r}_k}{\partial q_\alpha} = Q_\alpha, \text{ The generalised force corresponding to } q_\alpha.$$

Putting the values of the two sides of equation (4) we get,

$$\boxed{\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha} = Q_\alpha} \quad \text{--- (5)}$$

This is the general form of Lagrange's Equation.

If the system is conservative, i.e. when the force is defined by $\vec{F}_k = - \frac{\partial V_k}{\partial \vec{r}_k}$

V_k being the Potential energy of k th Particle, we get,

$$\begin{aligned}
 Q_\alpha &= \sum_k \vec{F}_k \cdot \frac{\partial \vec{r}_k}{\partial q_\alpha} \\
 &= \sum_k - \frac{\partial V_k}{\partial \vec{r}_k} \cdot \frac{\partial \vec{r}_k}{\partial q_\alpha} \\
 &= \sum_k - \frac{\partial V_k}{\partial q_\alpha}
 \end{aligned}$$

$$\therefore Q_x = - \frac{\partial}{\partial \dot{q}_x} \sum_k V_k$$

$= - \frac{\partial V}{\partial \dot{q}_x}$, when $V = \sum_k V_k$ is the total of the system.

Putting this value in equation (5) we get,

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_x} - \frac{\partial T}{\partial q_x} = - \frac{\partial V}{\partial q_x}$$

$$\text{i.e. } \frac{d}{dt} \frac{\partial}{\partial \dot{q}_x} (T-V) - \frac{\partial}{\partial q_x} (T-V) = 0$$

[As V is independent of \dot{q}_x
so, $\frac{\partial V}{\partial \dot{q}_x} = 0$].

or,

$$\boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_x} - \frac{\partial L}{\partial q_x} = 0} \quad \text{--- (6)}$$

As $(T-V) = L$, the Lagrangian of the system.

This is Lagrange's equation for the conservative system.