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PG - Sem II

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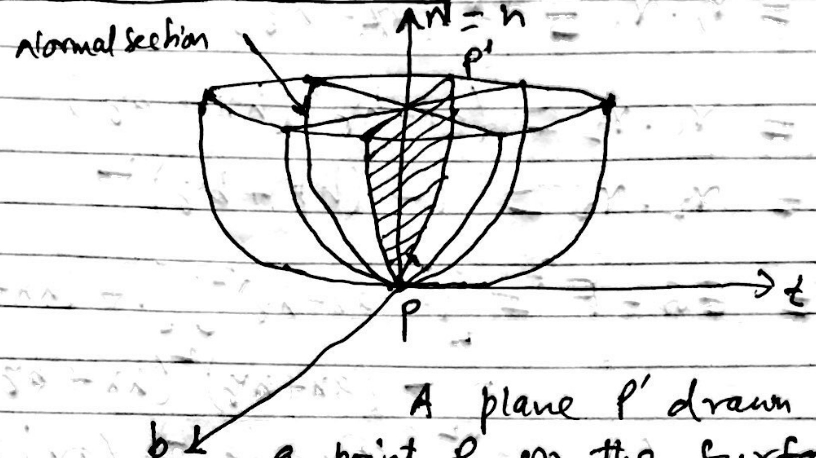
Paper - C-205
18.4.2020

Non-intrinsic properties of a Surface

NORMAL CURVATURE



A plane P' drawn through a point P on the surface, cuts the surface in a curve, it is called a section of the surface. If the plane P' contains the normal to the surface, then the curve is called Normal section. The principal normal n to the normal section is parallel to the surface normal N. By convention, we adopt $n = N$.

Let K_n represents the curvature of the normal section. Then K_n is called the normal curvature.

Note: If the plane P' does not contain the normal, then the section is called oblique section.

Qus ① Find the formula for normal curvature.

Ans: Let Eq. of surface is $r = r(u, v)$, where u, v are parameters. Let K_n represents the normal curvature at any point $P(u, v)$ to the surface.

Now $\frac{dt}{ds} = t' = r'' = K_n n$ (By S-F formula)

$= K_n N \quad (\because n = N)$

$\therefore \quad K_n N = r''$

$\Rightarrow \quad K_n N \cdot N = N \cdot r''$

$\Rightarrow \quad K_n = N \cdot r'' \qquad (N \cdot N = 1)$ —— ①

Again, $r' = \frac{dr}{ds} = \frac{\partial r}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial r}{\partial v}\frac{\partial v}{\partial s}$

$= r_1 u' + r_2 v'$

$$r'' = r_1 u'' + \frac{dr_1}{ds} u' + \frac{dr_2}{ds} v' + r_2 v''$$

$$= r_1 u'' + r_2 v'' + \left(\frac{\partial r_1}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial r_1}{\partial v}\frac{\partial v}{\partial s}\right)u' + \left(\frac{\partial r_2}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial r_2}{\partial v}\frac{\partial v}{\partial s}\right)v'$$

$$= r_1 u'' + r_2 v'' + r_{11} u'^2 + r_{12} u'v' + r_{21} u'v' + r_{22} v'^2$$

$$= r_1 u'' + r_2 v'' + r_{11} u'^2 + 2r_{12} u'v' + r_{22} v'^2 \quad \text{--- (2)}$$

$(\because r_{12} = r_{21})$

Now by (1)

$$K_n = N \cdot r''$$

$$= N \cdot (r_1 u'' + r_2 v'' + r_{11} u'^2 + 2r_{12} u'v' + r_{22} v'^2)$$ By (2)

$$= N \cdot r_1 u'' + N \cdot r_2 v'' + N \cdot r_{11} u'^2 + 2 N \cdot r_{12} u'v' + N \cdot r_{22} v'^2$$

$$= 0 + 0 + L u'^2 + 2M u'v' + N v'^2$$

$(\because r_1 \cdot N = 0,\ r_2 \cdot N = 0,\ r_{11} \cdot N = L,\ r_{12} \cdot N = M,\ r_{22} \cdot N = N)$

$$\therefore K_n = N \cdot r'' = L u'^2 + 2M u'v' + N v'^2$$

$$= L\left(\frac{du}{ds}\right)^2 + 2M\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + N\left(\frac{dv}{ds}\right)^2$$

$$= \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{ds^2}$$

$$K_n = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2} \quad \text{--- (3)}$$

(3) is called curvature of the normal section

Note: Normal curvature is denoted by K_n.
its formula is $K_n = N \cdot r''$, where r'' is curvature vector at any point P on the surface

Qus (2) Prove that curvature at any point P of normal section is equal to the normal curvature at P in same direction.

Ans let N be the unit normal vector to the surface at P. Then normal curvature K_n is given by

$$K_n = N \cdot r''.$$

Let K be the curvature at P of the normal section in the direction (du, dv).

Then $r'' = K_n = KN \qquad (\because n = N)$

$N \cdot r'' = N \cdot (\kappa N) = \kappa \qquad (\because N \cdot N = 1)$

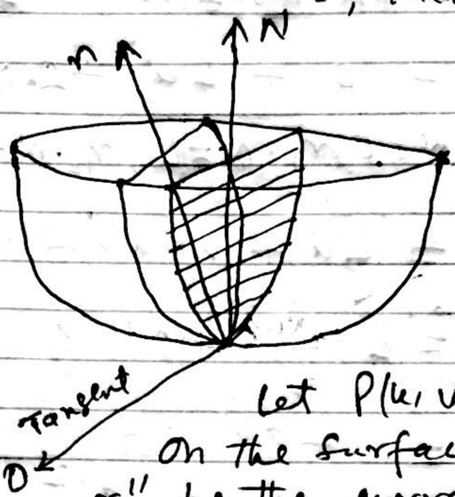
$\Rightarrow \kappa_n = \kappa$

Hence curvature at P of normal section containing the direction (du, dv) is equal to the normal curvature at P in the same direction.

Q No. 3 State and Prove Meusnier's theorem.

Statement: If κ and κ_n are the curvatures of oblique and normal sections through the same tangent line and θ be the angle between these sections, then $\kappa_n = \kappa \cos\theta$.

Proof



Let $P(u, v)$ be a point on the surface $r = r(u, v)$ and r'' be the curvature vector at P of the oblique section through P, containing the direction (du, dv). Then $r'' = \kappa n$ — (1)
where n is the unit principal normal vector to the oblique section at P.

N is the unit normal vector to the normal section of the surface at P in the same direction parallel to (du, dv). Since θ is the angle between the oblique and normal sections at P through the same tangent line, so θ is the angle between oblique and normal sections at P through the same the same tangent line. ie θ is the angle between the vectors n and N.

$\therefore \; n \cdot N = |n||N|\cos\theta = \cos\theta \qquad (\because |n| = 1, \; |N| = 1)$

Now taking the dot product of both sides of (1)

ie $r'' \cdot N = \kappa n \cdot N = \kappa \cos\theta$

Again $K_n = r''.N$ = normal curvature at P in the direction (du, dv) = curvature of the normal section at P parallel to the direction (du, dv).

$$\therefore \quad K_n = K\cos\theta$$

Hence Proved.

Definition (1) Principal Directions. The maximum or minimum curvatures at the points to the normal sections of a surface are called principal sections of the surface and the tangents to these sections at the point are called principal directions. There are two principal directions at every point on a surface and they are mutually orthogonal generally.

Definition (2) Principal curvature The maximum and minimum curvatures at a point are called principal curvatures at that point and their corresponding radius of curvatures are called principal radius of curvatures.

Q No (4) Find the equation of the giving principal curvatures.

Ans: Let K_n be the normal curvature at point $P(u, v)$ in the direction (du, dv). Then

$$K_n = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}$$

Let actual direction coefficients of (du, dv) is (l, m)

Then $$K_n = \frac{Ll^2 + 2Mlm + Nm^2}{El^2 + 2Flm + Gm^2},$$

where $El^2 + 2Flm + Gm^2 = 1$ ——— (1)

$$\therefore \quad K_n = Ll^2 + 2Mlm + Nm^2 \quad \text{——— (2)}$$

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L, M, N are fixed at P. So the value of κ_n depends upon the values of l, m at P.

So the κ_n is a function of two variables l, m and connected by relation (2).

Now we shall find the maximum value of κ_n. using Legarange's method of undetermined multipiers. For max^m or min^m value of κ_n, we have $d\kappa_n = 0$

ie $d(Ll^2 + 2Mlm + Nm^2) = 0$

$\Rightarrow L \cdot 2l\,dl + 2M(l\,dm + m\,dl) + N \cdot 2m\,dm = 0$

$\Rightarrow (Ll + Mm)dl + (Ml + Nm)dm = 0$ —— (3)

Taking derivative of (1), we have

$E \cdot 2l\,dl + 2F(l\,dm + m\,dl) + G \cdot 2m\,dm = 0$

$\Rightarrow (El + Fm)dl + (Fl + Gm)dm = 0$ —— (4)

Eq (4) $\times \lambda$ + Eq (3) $\Rightarrow$

$\{(Ll + Mm) + \lambda(El + Fm)\}dl + \{(Ml + Nm) + \lambda(Fl + Gm)\}dm = 0$

Equating to zero the coefficients of dl and dm, we have

$(Ll + Mm) + \lambda(El + Fm) = 0$ —— (5)

$(Ml + Nm) + \lambda(Fl + Gm) = 0$ —— (6)

Now Eq (5) $\times l$ + Eq (6) $\times m$ $\Rightarrow$

$(Ll^2 + 2Mlm + Nm^2) + \lambda(El^2 + 2Flm + Gm^2) = 0$

$\Rightarrow \kappa_n + \lambda \cdot 1 = 0 \Rightarrow \lambda = -\kappa_n$

Putting the value of λ in (5) and (6), we have

$(Ll + Mm) - \kappa_n(El + Fm) = 0$ —— (7)

$(Ml + Nm) - \kappa_n(Fl + Gm) = 0$ —— (8)

Now by (7), we have

$(L - \kappa_n E)l + (M - \kappa_n F)m = 0$ —— (9)

Now by (8), we have

$(M - \kappa_n F)l + (N - \kappa_n G)m = 0$ —— (10)

Now by (9), we have

$$(L - K_n E) l = -(M - K_n F) m \quad \text{—— (11)}$$

Now by (10), we have

$$(N - K_n G) m = -(M - K_n F) l \quad \text{—— (12)}$$

Eq (11) x Eq (12) $\Rightarrow$

$$(L - K_n E)(N - K_n G)\cancel{lm} = (M - K_n F)(M - K_n F)\cancel{lm}$$

$$\Rightarrow LN - K_n LG - K_n EN + K_n^2 EG$$
$$= M^2 - 2K_n FM + K_n^2 F^2$$

$$\Rightarrow K_n^2 (EG - F^2) - K_n (EN + LG - 2FM) + (LN - M^2) = 0 \quad \text{—— (12)}$$

Equation (12) is the required quadratic equation for giving the maximum or minimum values of normal curvature at P. Its roots are usually denoted by K_a and K_b.

Now sum of the roots of Eq (12)

ie $\quad K_a + K_b = \dfrac{EN + LG - 2FM}{EG - F^2}$

and Product of the roots of Eq (12)

ie $\quad K_a \cdot K_b = \dfrac{LN - M^2}{EG - F^2} = \dfrac{T^2}{H^2}$.

Note (1) First Curvature: The sum of principal curvatures K_a and K_b is called first curvature and it is denoted by J ie $J = K_a + K_b = \dfrac{EN + LG - 2FM}{EG - F^2}$

(2) Gaussian Curvature: The product of principal curvatures ie $K_a \cdot K_b$ is called Gaussian Curvature and it is denoted by K.

ie $K = K_a \cdot K_b = \dfrac{T^2}{H^2}$.

S. Asha [signature] 18/4/2020

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