We shall adopt a similar method of indicating boundary condition in two or three dimensional heat transfer problems.

An improtant note. In what follows, we have explained methods of solving boundary valve problems in some articles and indicated working rules for doing problems. You can solve a problem in one of the two following ways :

Method I. Prove the result of the relevant article completely. Then, compare the given problem with standard boundary value problem and use the relevant working rule as the case may be.

Method II. Proceed exactly as explained in the relevant article without using any result.

We have solved boundary value problem by both of the above two methods to make students familiar with the technique of solving boundary value problem. In examination, you can use any one of the above two methods.

PART I: PROBLEMS BASED ON ONE DIMENSIONAL HEAT (OR DIFFUSION) EQUATION

Situation I. ***When ends of the rod are kept at zero temperature.*** (3 cond^n)

2.2 A. *General solution of heat flow equation* $k(\partial^2 u/\partial x^2) = \partial u/\partial t$ *by the method of separation of variables.* **[Delhi Maths (H) 1997; Meerut 1995; Ravishankar 2002]**

Sol. Given $\qquad k(\partial^2 u/\partial x^2) = \partial u/\partial t$ ... (1)

Let solution of (1) be of the form $\qquad u(x, t) = X(x)\,T(t)$ ... (2)

We then find, on substituting in (1), that

$$kX''T = XT' \quad \text{or} \quad X''/X = T'/kT, \qquad \text{... (3)}$$

where the dashes denote derivatives with respect to the relevant variable. Clearly the L.H.S. of (3) is a function of x alone and the R.H.S. is a function of t alone. Since x and t are independent variables, (3) can hold good if each side is equal to a constant, say μ. Then (3) leads to

$$X'' - \mu X = 0 \qquad \text{... (4)}$$

and

$$T' = \mu kT \qquad \text{... (5)}$$

Three cases arise according as μ is zero, positive or negative.

Case I. Let $\mu = 0$. Then solutions of (4) and (5) are

$$X = a_1 x + a_2 \quad \text{and} \quad T = a_3 \qquad \text{... (6)}$$

Case II. Let μ *be* $+$ *ve*, say λ^2, where $\lambda \neq 0$. Then (4) and (5) become respectively $X'' - \lambda^2 X = 0$ and $T' = \lambda^2 kT$. Solving these differential equations, we obtain

$$X = b_1 e^{\lambda x} + b_2 e^{-\lambda x} \quad \text{and} \quad T = b_3 e^{\lambda^2 kt} \qquad \text{... (7)}$$

Case III. Let μ be $-$ ve, say $-\lambda^2$, where $\lambda \neq 0$. Then (4) and (5) become respectively $X'' + \lambda^2 X = 0$ and $T' = -\lambda^2 kT$. Solving these, we get

$$X = c_1 \cos \lambda x + c_2 \sin \lambda x \quad \text{and} \quad T = c_3 e^{-\lambda^2 kt} \qquad \text{... (8)}$$

Thus the various possible solutions are

$$u(x, t) = A_1 x + A_2, \qquad \text{... (9)}$$

$$u(x,t) = (B_1 e^{\lambda x} + B_2 e^{-\lambda x}) e^{\lambda^2 kt} \qquad \text{... (10)}$$

and

Imp

$$u(x,t) = (C_1 \cos \lambda x + C_2 \sin \lambda x) e^{-\lambda^2 kt}, \qquad \text{... (11)}$$

where $A_1 = a_1 a_3$, $A_2 = a_2 a_3$, $B_1 = b_1 b_3$, $B_2 = b_2 b_3$, $C_1 = c_1 c_3$ and $C_2 = c_2 c_3$ are new arbitrary constants.

Now we have to choose that solution which is consistent with the physical nature of the problem. Since we are dealing with problem of heat conduction, temperature $u(x, t)$ must decrease with the increase of time. Accordingly the solution given by (11) is the only suitable solution.

2.2 B Solved examples based on Art. 2.2 A

Example 1. *Explain the method of separation of variables for finding solutions of ... order linear partial differential equations. Hence, find the solution of one-dimensional diff... equation $\partial^2 z/\partial x^2 = (1/k)(\partial z/\partial t)$ which tends to zero as $t \to \infty$.*

[Kanpur ...]

Sol. ***Method of separation of variables.*** A powerful method of solving the following ... order linear partial differential equation can be used in certain circumstances

$$Rr + Ss + Tt + Pp + Qq + Zz = F,$$

where $r = \partial^2 z/\partial x^2, s = \partial^2 z/\partial x \partial y, t = \partial^2 z/\partial y^2$, $p = \partial z/\partial x, q = \partial z/\partial y$ and R, S, T, P, Q, Z ... are functions of x and y.

Let (*i*) possess a solution of the form $\quad z = X(x)\, Y(y)$,
where X is a function of x alone and Y is a function of y alone.

Substituting the above value of z in (*i*), (*i*) reduces to

$$(1/X)\, f(D)\, X = (1/Y)\, g(D')\; Y,$$

where $f(D)$, $g(D')$ are quadratic functions of $D(=\partial/\partial x)$ and $D'(=\partial/\partial y)$, respectively. Whe... reduces to (*iii*), we say that the equation (*i*) is separable in the variables x, y.

Now, the L.H.S. of (*iii*) is a function of x alone whereas the R.H.S. is a function of y a... and the two can be equal only if each is equal to a constant, λ say. Then (*iii*) gives the follow... pair of second-order linear ordinary differential equations.

$$f(D)X = \lambda X \qquad \text{and} \qquad g(D')Y = \lambda Y.$$

The problem of finding solutions of the form (ii) of (i) reduces to solving two equa... given by (iv).

To find the required solution of the given diffusion equation.

Proceed exactly as in Art. 2.2 A upto equations (11). Then proceed as follows.

We have three possible solutions, namely, (9), (10) and (11). The given condition dema... that $u \to 0$ as $t \to \infty$ we therefore reject the solutions given by (8) and (10). Hence, the des... solution is given by (11).

Ex. 2. *Use the method of separation of variables to solve the equation $\partial^2 v/\partial x^2 = \partial$... Given that $v = 0$ when $t \to \infty$ as well as $v = 0$ at $x = 0$ and $x = l$.*

[Delhi Maths (Hons) 2002, Delhi Maths (Hons) Physics 20... Lucknow UP (Tech) 2005; Meerut 2000, ...

Sol. Proceed as in Art. 2.2 A by noting that here $u = v$, $k = 1$ upto equation (11). Thus ... obtain the following three possible types of solutions of $\partial^2 v/\partial x^2 = \partial v/\partial t$.

$$v(x, t) = A_1 x + A_2$$

$$v(x, t) = (B_1 e^{\lambda x} + B_2 e^{-\lambda x}) e^{\lambda^2 t}, \lambda \neq 0$$

$$v(x, t) = (C_1 \cos \lambda x + C_2 \sin \lambda x) e^{-\lambda^2 t}, \lambda \neq 0$$

We are given that $v \to 0$ as $t \to \infty$. We, therefore, reject the solutions given by (9)' ... (10)'. Hence, the desired solution is given by (11)', namely,

$$v(x,t) = (C_1 \cos \lambda x + C_2 \sin \lambda x) e^{-\lambda^2 t}, \lambda \neq 0$$

The given boundary conditions are

$$v(0, t) = 0$$

and

$$v(l, t) = 0$$

Imp $\quad \boxed{V(x, \infty) = 0}$

Putting $x = 0$ in (12) and using B.C. (13), we obtain $C_1 = 0$ Then, (12) reduces to

$$v(x,t) = C_2 \sin \lambda x \, e^{-\lambda^2 t} \qquad \text{... (15)}$$

Putting $x = l$ in (15) and using B.C. (14), we obtain

$0 = C_2 \sin \lambda l \, e^{-\lambda^2 t}$ giving $C_2 \sin \lambda l = 0$ = sin nπ ... (16)

Since we are looking for a non-trivial solution, we take $C_2 \neq 0$. Hence (16) reduces to λ = nπ/l

$\sin \lambda l = 0$ giving $\lambda l = n\pi$ so that $\lambda = n\pi / l, \; n = 1, 2, 3, ...$ ... (17)

Hence, from (11)′ a solution $v_n(x, t)$ of the given equation for some value of n is given by

$$v_n(x,t) = B_n \sin(n\pi x / l)\, e^{-(n^2\pi^2 t)/l^2}, \text{ by setting } C_2 = B_n. \qquad \text{... (18)}$$

Noting that the given equation $\partial^2 v / \partial x^2 = \partial v / \partial t$ is linear, its most general solution is obtained by applying the principle of superposition. Thus, we have

$$v(x,t) = \sum_{n=1}^{\infty} v_n(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}\, e^{-(n^2\pi^2 t)/l^2}$$

2.3A. General solution of heat equation when both the ends of a bar are kept at temperature zero and the initial temperature is prescribed. (3 condn)

If both the ends of a bar of length a are at temperature zero and the initial temperature is to be prescribed function f(x) in the bar, then find the temperature at a subsequent time t.

or The faces x = 0 and x = a of an infinite slab are maintained at zero temperature. Given that the temperature u (x, t) = f(x) at t = 0. Determine the temperature at a subsequent time t.

[Meerut 2010; Delhi B.A./B.Sc. (Hons) III 2012; Agra 2004; Aligarg 2003; Andhra 2003]

Sol. Here the temperature $u(x, t)$ in the given solid is governed by the one dimensional heat equation

$$k(\partial^2 u / \partial x^2) = \partial u / \partial t \qquad \text{... (1)}$$

Since the ends $x = 0$ and $x = a$ are kept at zero temperature, the boundary conditions are IMP

$$\boxed{u(0, t) = 0} \quad \text{and} \quad u(a, t) = 0, \quad \text{for all } t \qquad \text{... (2)}$$

The initial condition is given by $u(x, 0) = f(x),\; 0 < x < a$ ... (3)

Suppose that (1) has solutions of the form $u(x, t) = X(x)\, T(t)$ ... (4)

where X is a function of x alone and T that of t alone.

Substituting this value of u in (1), we get

$$k X'' T = X T' \quad \text{or} \quad X''/X = T'/kT \qquad \text{... (5)}$$

Since x and t are independents variables, (5) can only be true if each side is equal to the same constant, say μ. Hence (5) leads to the following equations: ... (6)

$$X'' - \mu X = 0 \qquad \text{...(7)}$$

and

$$T' = \mu k T. \qquad \text{... (8)}$$

Using (2), (4) gives $X(0)\, T(t) = 0$ and $X(a)\, T(t) = 0$

Since $T(t) = 0$ leads to $u = 0$, so suppose that $T(t) \neq 0$.

∴ From (8), $X(0) = 0$ and $X(a) = 0$ ... (9)

We now solve (6) under B.C. (9). Three cases arise.