Definition: Infinite fourier Cosine transform:

Let $F(x)$ be a function defined on $(-\infty, \infty)$ and be piecewise continuous in each partial interval and absolutely integrable in $(-\infty, \infty)$. Then the infinite fourier cosine transform of $F(x)$ is a function of a new variable s and it is denoted and defined as

$$F_c\{F(x)\} = \overline{F_c}(s) = f_c(s) = \int_0^\infty F(x) \cos sx \, dx.$$

The function $F(x)$ is called inverse fourier cosine transform of $f_c(s)$ and is denoted by

$$F(x) = F_c^{-1}\{f_c(s)\}$$

Inversion formula for inverse fourier Cosine transform

If $f_c(s)$ is the infinite fourier cosine transform of $F(x)$ and if $F(x)$ satisfies the Dirichlet condition in every finite interval $(-l, l)$ and further if $F(x)$ is absolutely integrable in $(-\infty, \infty)$, then at every point of continuity of $F(x)$,

$$F(x) = \frac{2}{\pi} \int_0^\infty f_c(s) \cos sx \, ds.$$

Note: Dirichlet's Conditions Let $f(x)$ satisfy the following conditions.

(i) $f(x)$ is defined in the interval $-l < x < l$

(ii) $f(x)$ and $f'(x)$ are sectionally continuous in $(-l, l)$

(iii) $f(x)$ is periodic with period $2l$.

Then the above conditions are called Dirichlet's conditions.

Relationship between Fourier transform and Laplace transform: We define a function $F(t)$ as follows:

$$F(t) = \begin{cases} e^{-xt}\phi(t), & t > 0 \\ 0, & t < 0 \end{cases}$$

$$F\{F(t)\} = \int_{-\infty}^{\infty} e^{-iyt} F(t)\,dt$$

$$= \int_{-\infty}^{0} e^{-iyt} F(t)\,dt + \int_{0}^{\infty} e^{-iyt} F(t)\,dt$$

$$= \int_{-\infty}^{0} e^{-iyt} \cdot 0\,dt + \int_{0}^{\infty} e^{-iyt} e^{-xt}\phi(t)\,dt$$

$$= \int_{0}^{\infty} e^{-(x+iy)t} \cdot \phi(t)\,dt$$

$$= \int_{0}^{\infty} e^{-st}\phi(t)\,dt, \text{ where } x+iy = s$$

$$= L\{\phi(t)\}.$$

$$\therefore F\{F(t)\} = L\{\phi(t)\}$$

This is the required relation between Fourier transform and Laplace transform.

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