Putting $x = 0$ in (12) and using B.C. (13), we obtain $C_1 = 0$ Then, (12) reduces to

$$v(x,t) = C_2 \sin \lambda x \, e^{-\lambda^2 t} \quad \text{... (15)}$$

Putting $x = l$ in (15) and using B.C. (14), we obtain

$0 = C_2 \sin \lambda l \, e^{-\lambda^2 t}$ giving $C_2 \sin \lambda l = 0$ = sin nπ ... (16)

λ = nπ/L

Since we are looking for a non-trivial solution, we take $C_2 \neq 0$. Hence (16) reduces to

$\sin \lambda l = 0$ giving $\lambda l = n\pi$ so that $\lambda = n\pi/l, \; n = 1, 2, 3, \ldots$ ... (17)

Hence, from (11)′ a solution $v_n(x, t)$ of the given equation for some value of n is given by

$$v_n(x,t) = B_n \sin(n\pi x/l)\, e^{-(n^2\pi^2 t)/l^2}, \text{ by setting } C_2 = B_n. \quad \text{... (18)}$$

Noting that the given equation $\partial^2 v/\partial x^2 = \partial v/\partial t$ is linear, its most general solution is obtained by applying the principle of superposition. Thus, we have

$$v(x,t) = \sum_{n=1}^{\infty} v_n(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} e^{-(n^2\pi^2 t)/l^2}$$

2.3A. General solution of heat equation when both the ends of a bar are kept at temperature zero and the initial temperature is prescribed.

(3 cond^n)

If both the ends of a bar of length a are at temperature zero and the initial temperature is to be prescribed function f(x) in the bar, then find the temperature at a subsequent time t.

Or The faces x = 0 and x = a of an infinite slab are maintained at zero temperature. Given that the temperature u (x, t) = f(x) at t = 0. Determine the temperature at a subsequent time t.

[Meerut 2010; Delhi B.A./B.Sc. (Hons) III 2012; Agra 2004; Aligarg 2003; Andhra 2003]

Sol. Here the temperature $u(x, t)$ in the given solid is governed by the one dimensional heat equation

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t \quad \text{... (1)}$$

Since the ends $x = 0$ and $x = a$ are kept at zero temperature, the boundary conditions are

IMP

$$u(0, t) = 0 \quad \text{and} \quad u(a, t) = 0, \quad \text{for all } t \quad \text{... (2)}$$

The initial condition is given by $u(x, 0) = f(x),\ 0 < x < a$ ... (3)

Suppose that (1) has solutions of the form $u(x, t) = X(x)\, T(t)$ ... (4)

where X is a function of x alone and T that of t alone.

Substituting this value of u in (1), we get

$$kX''T = XT' \quad \text{or} \quad X''/X = T'/kT \quad \text{... (5)}$$

Since x and t are independents variables, (5) can only be true if each side is equal to the same constant, say μ. Hence (5) leads to the following equations:

$$X'' - \mu X = 0 \quad \text{... (6)}$$

and

$$T' = \mu k T. \quad \text{...(7)}$$

Using (2), (4) gives $X(0)\,T(t) = 0$ and $X(a)\,T(t) = 0$ ... (8)

Since $T(t) = 0$ leads to $u = 0$, so suppose that $T(t) \neq 0$.

$\therefore$ From (8), $X(0) = 0$ and $X(a) = 0$ ... (9)

We now solve (6) under B.C. (9). Three cases arise.

Case I. Let $\mu = 0$. Then solution of (6) is $X(x) = Ax + B$

Using B.C. (9), (10) gives $0 = B$, $0 = Aa + B$ so that $A = B = 0$.

Hence $X(x) \equiv 0$ so that $u \equiv 0$, which does not satisfy (3). So we reject $\mu = 0$

Case II. Let $\mu = \lambda^2, \lambda \neq 0$. Then solution of (6) is $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$

Using B C. (9), (11) gives $0 = A + B$ and $0 = Ae^{a\lambda} + Be^{-a\lambda}$

Solving (12), $A = B = 0$ so that $X(x) \equiv 0$ and hence $u = 0$,

which does not satisfy (3). So we also reject $\mu = \lambda^2$.

Case III. Let $\mu = -\lambda^2, \lambda \neq 0$, Then solution of (6) is $X(x) = A\cos\lambda x + B\sin\lambda x$

Using B.C. (9), (13) gives $0 = A$ and $0 = A\cos\lambda a + B\sin\lambda a$

So $\sin\lambda a = 0$. We have taken $B \neq 0$, since otherwise $X \equiv 0$ so that $u \equiv 0$ which does not satisfy (3). Solving the trignometric equation $\sin\lambda a = 0$, we have

$$\lambda a = n\pi \quad \text{so that} \quad \lambda = n\pi / a, \quad \text{where} \quad n = 1, 2, ...$$

Hence non-zero solutions $X_n(x)$ of (6) are given by $X_n(x) = B_n \sin(n\pi x / a)$

Using (14), (7) reduces to $\dfrac{dT}{T} = -\dfrac{n^2\pi^2 k}{a^2}\, dt$, as $\mu = -\lambda^2 = -\dfrac{n^2\pi^2}{a^2}$

or $(1/T)\, dT = -C_n^2 dt$, where $C_n^2 = (n^2\pi^2 k)/a^2$

whose general solution is $T_n(t) = D_n e^{-C_n^2 t}$

$$\therefore \quad \boxed{u_n(x,t) = X_n(x) T_n(t) = E_n \sin(n\pi x / a) e^{-C_n^2 t}}$$

are solutions of (1), satisfying (2). Here $E_n (= B_n D_n)$ is another arbitrary constant. In order to obtain a solution also satisfying (3), we consider more general solution.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin(n\pi x / a) e^{-C_n^2 t}.$$

Substituting $t = 0$ in (18) and using (3), we get $f(x) = \sum_{n=1}^{\infty} E_n \sin(n\pi x / a)$

which is Fourier sine series. So the constants E_n are given by

$$E_n = \frac{2}{a}\int_0^a f(x)\sin(n\pi x / a)\, dx, \qquad n = 1, 2, 3$$

Hence (18) is the required solution where E_n is given by (19).

2.3 B. Working rule for solving heat equation when both the ends of a bar of length are kept at temperature zero and the initial temperature $f(x)$ is prescribed.

Step I. Proceed as in Art. 2.3 A and prove that the solution of the heat equation

$$k(\partial^2 u / \partial x^2) = \partial u / \partial t$$

subject to the boundary conditions $u(0, t) = u(a, t) = 0$, for all t

and the initial condition $u(x, 0) = f(x)$, $0 < x < a$

is given by

$$u(x,t) = \sum_{n=0}^{\infty} E_n \sin(n\pi x / a) e^{-C_n^2 t}$$