Th^m: Let a, b, c be any 3 elements of euclidean ring R and $(a,b)=1$. ie g.c.d of a & b is 1. If $a|bc$ then $a|c$.

Proof Since g.c.d of a & b is 1.

$\Rightarrow \exists\ \lambda, \mu \in R$ such that $1 = \lambda a + \mu b$ —— ①

Multiplying both sides with c.

$\Rightarrow c = \lambda ac + \mu bc$ —— ②

Since $a|bc \Rightarrow \exists\ q \in R$ such that

$bc = aq$ —— ③

Putting in ② we get

$c = \lambda ac + \mu qa$

$\Rightarrow c = (\lambda c + \mu q)a$

$\Rightarrow a | c$ proved

Th^m If p is prime element in Euclidean ring R and $p|ab$ where $a, b \in R$ then p divides at least one of a & b.

Proof! If $p|a$ then we have nothing to prove so let $p \nmid a$. Since p is prime and $p \nmid a$ $\Rightarrow$ gcd of p & $a = 1$. also $p|ab$ so by previous th^m $p|b$. Proved

Thm Let R be a Euclidean ring. Let $a, b \in R$ be non-zero elements. Then

(I) If b is unit in R then $d(ab) = d(a)$

(II) If b is not a unit in R than $d(ab) > d(a)$

Proof: (I) We know that in a Euclidean ring

$$d(ab) \geq d(a) \quad \text{——— (1)}$$

Let b is unit in R

$\Rightarrow$ b is invertible i.e $b^{-1} \in R$

So, We can write

$$a = (ab)b^{-1}$$

$$\Rightarrow d(a) = d[(ab)b^{-1}] \geq d(ab)$$

$$\Rightarrow d(a) \geq d(ab) \quad \text{——— (2)}$$

From (1) & (2)

$$d(ab) = d(a)$$

(II) If b is not a unit in R

Since a & b are non-zero elements of R

$\Rightarrow ab$ is non-zero element of R

So by division algorithm $\exists\, q, r \in R$ such that

$$a = (ab)q + r$$

Where either $r = 0$ or $d(r) < d(ab)$ ——— (3)

Now, if $r = 0$

$$\Rightarrow a = (ab)q$$

$\Rightarrow a - abq = 0$

$\Rightarrow a(1-bq) = 0$

$\Rightarrow 1 - bq = 0$ ($\because a \neq 0$ & R has no divisor of zero)

$\Rightarrow bq = 1.$

$\Rightarrow b$ is invertible

$\Rightarrow b$ is a unit in R. which is a contradiction

Hence r cannot be zero

$\Rightarrow d(r) < d(ab)$ [from (3)]

ie $d(ab) > d(r)$ —— (4)

Also from (3)

$r = a - abq$

$\Rightarrow d(r) = d[a(1-bq)] \geq d(a)$

ie $d(r) \geq d(a)$ —— (5)

From (4) & (5)

$d(ab) > d(a)$

Proved