IMPORTANT THEOREMS

Theorem 1. If c_1 and c_2 are arbitrary constants, then $F\{c_1 F(x) + c_2 G(x)\} = c_1 F\{F(x)\} + c_2 F\{G(x)\}$

(It is also called Linearity property)

Proof: $F\{c_1 F(x) + c_2 G(x)\} = \int_{-\infty}^{\infty} e^{-isx}\{c_1 F(x) + c_2 G(x)\}\, dx$

$$= \int_{-\infty}^{\infty} \{ c_1 e^{-isx} F(x) + c_2 e^{-isx} G(x)\}\, dx$$

$$= \int_{-\infty}^{\infty} c_1 e^{-isx} F(x)\, dx + \int_{-\infty}^{\infty} c_2 e^{-isx} G(x)\, dx$$

$$= c_1 \int_{-\infty}^{\infty} e^{-isx} F(x)\, dx + c_2 \int_{-\infty}^{\infty} e^{-isx} G(x)\, dx$$

$$= c_1 F\{F(x)\} + c_2 F\{G(x)\}$$

Proved

Theorem 2: If $f(s)$ is the fourier transform of $F(x)$, then $\frac{1}{a} f\left(\frac{s}{a}\right)$ is the fourier transform of $F(ax)$.

(Change of scale property).

Proof: $\because F\{F(x)\} = \int_{-\infty}^{\infty} e^{-isx} F(x)\, dx = f(s).$

$\therefore F\{F(ax)\} = \int_{-\infty}^{\infty} e^{-isx} F(ax)\, dx$

$$= \int_{-\infty}^{\infty} e^{-\frac{ist}{a}} F(t) \frac{dt}{a}, \text{ where } ax = t$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} e^{-i\left(\frac{s}{a}\right)t} F(t)\, dt = \frac{1}{a} f\left(\frac{s}{a}\right)$$

Proved.

Theorem 3: If $f_c(s)$ is the fourier cosine transform of $F(x)$, then show that Fourier cosine transform of $F(\frac{x}{a})$ is $a f_c(as)$.

Proof:

$$F_c\left\{F\left(\frac{x}{a}\right)\right\} = \int_0^{\infty} F\left(\frac{x}{a}\right) \cos sx \, dx$$

Put $\frac{x}{a} = t \Rightarrow x = at \Rightarrow dx = a\,dt$

$$= \int_0^{\infty} F(t) \cos(sat) . a\,dt$$

$$= \int_0^{\infty} a\,F(t) \cos(sat)\,dt$$

$$= a \int_0^{\infty} F(t) \cos(sat)\,dt = a\,f_c(as).$$

Proved.

Theorem 4: If $f_s(s)$ is the fourier sine transform of $F(x)$, then show that fourier sine transform of $F(\frac{x}{a})$ is $a f_s(as)$.

Proof:

$$f_s\left\{F\left(\frac{x}{a}\right)\right\} = \int_0^{\infty} F\left(\frac{x}{a}\right) \sin sx \, dx$$

Put $\frac{x}{a} = t \Rightarrow x = at \Rightarrow dx = a\,dt$

$$= \int_0^{\infty} F(t) \sin(sat) . a\,dt$$

$$= a\int_0^{\infty} F(t) \sin(ast)\,dt$$

$$= a\,f_s(as).$$ Proved.

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Theorem 5: If $f(s)$ is the Fourier transform of $F(x)$, then prove that $e^{-ias} f(s)$ is the fourier transform of $F(x-a)$. (Shifting Property)

Proof: $$F\{F(x-a)\} = \int_{-\infty}^{\infty} e^{-isx} F(x-a)\, dx$$

Put $x-a = t \Rightarrow dx = dt$

$$= \int_{-\infty}^{\infty} e^{-is(a+t)} F(t)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-ias} \cdot e^{-ist} F(t)\, dt$$

$$= e^{-ias} \int_{-\infty}^{\infty} e^{-ist} F(t)\, dt = e^{-ias} f(s)$$

Proved.

Theorem 6: If $F(x)$ has the fourier transform $f(s)$, then $F(x)\cos ax$ has the Fourier transform $\frac{1}{2} f(s-a) + \frac{1}{2} f(s+a)$. (Modulation theorem)

Proof: Since we know that $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$$F\{F(x)\cos(ax)\} = \int_{-\infty}^{\infty} e^{-isx} F(x) \cos ax\, dx$$

$$= \int_{-\infty}^{\infty} e^{-isx} F(x) \frac{e^{iax} + e^{-iax}}{2}\, dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{-i(s-a)x} F(x)\, dx + \int_{-\infty}^{\infty} e^{-i(s+a)x} F(x)\, dx\right]$$

$$= \frac{1}{2}\left[f(s-a) + f(s+a)\right]$$

Proved.

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