$$\boxed{E_n = \frac{2}{a}\int_0^a f(x)\sin(n\pi x/a)dx,\ n = 1, 2, 3, \ldots} \quad \ldots(5)$$

$$C_n^2 = (n^2\pi^2 k)/a^2 \quad \ldots(6)$$

Step II. Compare the given problem with (1), (2) and (3) and find particular values of ..., and $f(x)$.

Step III. Substitute the partiular values of k, a and $f(x)$ in (5) and (6) to get E_n and C_n^2

Step IV. Substitute the values of coefficients E_n and C_n^2 obtained is step III in (4) to arrive ... the desired solution of the given boundary value problem.

2.3C Solved example based on Art. 2.3 A and Art. 2.3 B

Ex. 1 (a) *A rod of length l with insulated sides, is initially at a uniform temperature u_0. Its ends are suddenly cooled to 0°C and are kept at that temperature. Find the temperation $u(x, t)$.*

[Andhra 1997, 2003]

(b) *A thin rod of length π is first immersed in boiling water so that its temperature is 100°C throughout. Then the rod is removed from water at $t = 0$ which is immediately put in ice so that the ends are kept at 0°C. Find $w(x, t)$ if heat equation is $a^2(\partial^2 w/\partial x^2) = \partial w/\partial t$.* **[Nagpur 2002]**

Sol. (a) We can prove (actually prove in examination) that the solution of the heat equation

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t \quad \ldots(1)$$

subject to the boundary conditions $\quad u(0, t) = u(a, t) = 0$, for all t. ... (2)

and the initial condition $\quad u(x, 0) = f(x),\ 0 < x < a$ ... (3)

is given by

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/a)\, e^{-C_n^2 t},\ n = 1,2,3,\ldots \quad \ldots(4)$$

where

$$E_n = \frac{2}{a}\int_0^a f(x)\sin(n\pi x/a)dx,\ n = 1, 2, 3 \quad \ldots(5)$$

and

$$C_n^2 = (n^2\pi^2 k)/a^2 \quad \ldots(6)$$

Comparing the given boundary value problem with the boundary value problem given by (1), (2) and (3), we have $k = k$, $a = l$ and $f(x) = u_0$. Hence, (5) reduces to

$$E_n = \frac{2}{l}\int_0^l u_0 \sin\frac{n\pi x}{l}dx = \frac{2u_0}{l}\left[-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right]_0^l = \frac{2u_0}{n\pi}[1-(-1)^n],\ \text{as}\ \cos n\pi = (-1)^n$$

$$= \begin{cases} 0, & \text{if } n = 2m, \quad m = 1,2,3,\ldots \\ 4u_0/n\pi, & \text{if } n = 2m-1, \quad m = 1,2,3,\ldots \end{cases}$$

Hence solution (4) reduces to

$$u(x,t) = \sum_{m=1}^{\infty} E_{2m-1}\sin\frac{(2m-1)\pi x}{l}e^{-C_{2m-1}^2 t} \quad \text{or} \quad u(x,t) = \frac{4u_0}{\pi}\sum_{m=1}^{\infty}\frac{l}{(2m-1)}\sin\frac{(2m-1)\pi x}{l}e^{-C_{2m-1}^2 t},$$

where

$$C_{2m-1}^2 = \{(2m-1)^2\pi^2 k\}/l^2.$$

(b) Proceed as in part (a). Here $w = u$, $k = a^2$, $a = \pi$ and $u_0 = 100$.

Then, by (6), $C_n^2 = (2m-1)^2 a^2$ and so from (4) the required solution is

$$\therefore \quad w(x,t) = \frac{400}{\pi}\sum_{m=1}^{\infty}\frac{1}{2m-1}\sin(2m-1)x\, e^{-(2m-1)^2 a^2 t}$$

Ex. 2(a) *Solve the boundary value problem* $\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t)$ *satisfying the cond* $u(0, t) = u(l, t) = 0$ and $u(x, 0) = lx - x^2$. **[Himanchal 2007; Meerut 2010; Delhi 20**

(*b*) *Solve the boundary value problem* $\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t)$ *satisfying the cond* $u(0, t) = u(l, t) = 0$ and $u(x, 0) = x$ when $0 \le x \le l/2$, $u(x, 0) = l - x$ when $l/2 \le x \le l$.

[Meerut 1998,

Sol. We can prove that the solution of heat equation

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t$$

subject to the boundary conditions $\quad u(0, t) = u(a, t) = 0$ for all t

and the initial condition $\quad u(x, 0) = f(x),\ 0 < x < a$

is given by
$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/a)e^{-C_n^2 t}$$

where
$$E_n = \frac{2}{a}\int_0^a f(x)\sin(n\pi x/a)dx;\ n = 1, 2, 3, ...$$

and
$$C_n^2 = (n^2\pi^2 k)/a^2$$

Part (*a*) : Comparing the given boundary value problem with the boundary value proble given by (1), (2) and (3), we have $k = k$, $a = l$ and $f(x) = lx - x^2$. Hence, (5) reduces to

$$E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left[(lx - x^2)\left\{\frac{-\cos(n\pi x)/l}{(n\pi)/l}\right\} - (l - 2x)\left\{\frac{-\sin(n\pi x)/l}{(n\pi)^2/l^2}\right\} + (-2)\left\{\frac{\cos(n\pi x)/l}{(n\pi)^3/l^3}\right\}\right]_0^l$$

[Using the chain rule of integration by pa

$$= (2/l)\{(-2l^3/n^3\pi^3)\cos n\pi + (2l^3/n^3\pi^3)\} = (4l^2/n^3\pi^3)\{1 - (-1)^n\}$$

$$\therefore \quad E_n = \begin{cases} (8l^2)/(2m-1)^3\pi^3, \text{ if } n = 2m - 1 \text{ (odd) and } m = 1, 2, 3, ... \\ 0, \text{ if } n = 2m \text{ (even) where } m = 1, 2, 3, ... \end{cases}$$

Then, by (6), $C_n^2 = \{(2m-1)^2\pi^2 k\}/l^2$ and so from (4) the required solution is given by

$$u(x,t) = \frac{8l^2}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}e^{-\{(2m-1)^2\pi^2 kt\}/l^2}$$

Part (b) : Comparing the given boundary value problem with the boundary value proble given by (1), (2) and (3), we have $k = k$, $a = l$ and

$$f(x) = \begin{cases} x, & \text{when } 0 \le x \le l/2 \\ l - x, & \text{when } l/2 \le x \le l \end{cases}$$

$$\therefore (5) \Rightarrow E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\left[\int_0^{l/2} f(x)\sin\frac{n\pi x}{l}dx + \int_{l/2}^l f(x)\sin\frac{n\pi x}{l}dx\right]$$

$$\ldots\frac{2x}{l}\sin\frac{n\pi x}{l}\,dx+\int_{l/2}^{l}\frac{2}{l}(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$\ldots\frac{2x}{l}\left(-\frac{\cos(n\pi x)/l}{(n\pi)/l}\right)-\left(\frac{2}{l}\right)\left(-\frac{\sin(n\pi x)}{(n\pi)^2/l^2}\right)\Bigg]_0^{l/2}+\left[\left(\frac{2(l-x)}{l}\right)\left(-\frac{\cos(n\pi x)/l}{(n\pi)/l}\right)-\left(-\frac{2}{l}\right)\left(-\frac{\sin(n\pi x)/l}{(n\pi)^2/l^2}\right)\right]_{l/2}^{l}$$

[Using chain rule of integration by parts]

$$\ldots n\pi)\cos(n\pi/2)+(2l/n^2\pi^2)\sin(n\pi/2)+(l/n\pi)\cos(n\pi/2)+(2l/n^2\pi^2)\sin(n\pi/2)$$

$$\therefore \quad E_n=\frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}=\begin{cases}0, \text{ if } n=2m \text{ and } m=1,2,3,\ldots\\ 4l/(2m-1)^2\pi^2, \text{ if } n=2m-1 \text{ and } m=1,2,3,\ldots\end{cases}$$

Then, $\quad (6) \Rightarrow C_n^2=\{(2m-1)^2\pi^2 k\}/l^2$

$\therefore$ From (4),
$$u(x,t)=\frac{4l}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\sin\frac{(2m-1)\pi x}{l}\,e^{-\{(2m-1)^2\pi^2 kt\}/l^2}$$

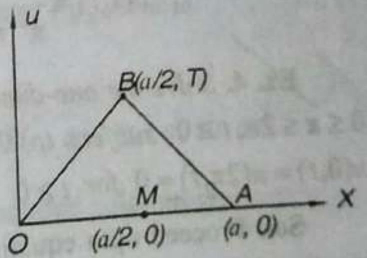
Ex. 2 (c) *Solve* $\partial u/\partial t=\partial^2 u/\partial x^2,\ 0<x<l,\ t>0$ *given that* $u(0,t)=u(l,t)=0$ *and* $u(x,0)=x(l-x),\ 0\le x\le l.$ **[I.A.S. 2002]**

Sol. Refer solved Ex. 2(*a*). Here $k=1$ and hence the solution reduces to

$$u(x,t)=\frac{8l^2}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}\,e^{-\{(2m-1)^2\pi^2 t/l^2}$$

Ex. 3. *A homogeneous rod of conducting material of length a has its ends kept at zero temperature. The temperature at the centre is T and falls uniformly to zero at the two ends. Find the temperature function u (x, t).*

Sol. We know that $u(x, t)$ is the solution of heat equation $\partial^2 u/\partial x^2=(l/k)(\partial u/\partial t)$. Here the boundary conditions are $u(0,t)=u(a,t)=0$ for all $t\ge 0$. Let OA be the given rod and M be its middle point. Given that the temperature at the centre M is T and falls uniformly to zero at the two ends O and A of the rod. Hence, the temperature distribution at $t=0$ is as given in the adjoining figure. The equations of straight lines OB and BA respectively are given by



$$u-0=\frac{T-0}{(a/2)-0}(x-0) \quad \text{and} \quad u-0=\frac{T-0}{(a/2)-a}(x-a) \qquad \ldots (i)$$

We can prove that the solution of heat equation
$$k(\partial^2 u/\partial x^2)=\partial u/\partial t \qquad \ldots (1)$$
subject to the boundary conditions $u(a,t)=u(a,t)=0,\quad$ for all $t \qquad \ldots (2)$

and the initial condition $u(x,0)=f(x),\quad 0<x<a \qquad \ldots (3)$

is given by
$$u(x,t)=\sum_{n=1}^{\infty}E_n\sin(n\pi x/a)^{-C_n^2 t} \qquad \ldots (4)$$

where
$$E_n=\frac{2}{a}\int_0^a f(x)\sin(n\pi x/a)\,dx,\ n=1,2,3, \qquad \ldots (5)$$

and
$$C_n^2=(n^2\pi^2 k)/a^2 \qquad \ldots (6)$$

Comparing the given boundary value problem with the boundary value problem given by (1), (2) and (3), we have $k=k,\ a=a$. Also, from (*i*), we have