

Fourier Transform (v)

Theorem (7) Derivative theorem: The Fourier transform of $F'(x)$, the derivative of $F(x)$, is $is f(s)$, where $f(s)$ is the Fourier transform of $F(x)$.

Proof: By definition, $F\{F'(x)\} = \int_{-\infty}^{\infty} e^{-isx} F'(x) dx$.

$$\begin{aligned} \text{ie } F\{F'(x)\} &= \left[e^{-isx} F(x) \right]_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} (-is) e^{-isx} F(x) dx \\ &= 0 + is \int_{-\infty}^{\infty} F(x) e^{-isx} dx \quad (\because F(x) \rightarrow 0 \text{ as } x \rightarrow \pm\infty) \\ &= is f(s) \end{aligned}$$

$\therefore F\{F'(x)\} = is f(s)$ if $F(x) \rightarrow 0$ as $x \rightarrow \pm\infty$.

Theorem (8) To show that

$$F\left\{\frac{d^n F(x)}{dx^n}\right\} = (is)^n f(s), \text{ where } F\{F(x)\} = f(s),$$

if the first $(n-1)$ derivatives of $F(x)$ vanish identically as $x \rightarrow \pm\infty$.

Proof: Let first $(n-1)$ derivatives of $F(x)$ vanish as $x \rightarrow \pm\infty$

$$\begin{aligned} F\left\{\frac{d^n F}{dx^n}\right\} &= \int_{-\infty}^{\infty} e^{-isx} \frac{d^n F}{dx^n} dx \quad \text{Integrating by parts} \\ &= \left[e^{-isx} \frac{d^{n-1} F}{dx^{n-1}} \right]_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} (-is) e^{-isx} \frac{d^{n-1} F}{dx^{n-1}} dx \\ &= 0 + is \int_{-\infty}^{\infty} e^{-isx} \frac{d^{n-1} F}{dx^{n-1}} dx = (is) F\left\{\frac{d^{n-1} F}{dx^{n-1}}\right\} \end{aligned}$$

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Again, $F\left\{\frac{d^n F}{dx^n}\right\} = is \int_{-\infty}^{\infty} e^{-isx} \frac{d^{n-1} F}{dx^{n-1}} dx$,
Integrating by parts, we have

$$\begin{aligned} F\left\{\frac{d^n F}{dx^n}\right\} &= (is) \left[\left[e^{-isx} \frac{d^{n-1} F}{dx^{n-1}} \right]_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} (-is) e^{-isx} \frac{d^{n-2} F}{dx^{n-2}} dx \right] \\ &= is \cdot 0 + (is)^2 \int_{-\infty}^{\infty} e^{-isx} \frac{d^{n-2} F}{dx^{n-2}} dx \\ &= (is)^2 F\left\{\frac{d^{n-2} F}{dx^{n-2}}\right\}. \end{aligned}$$

Similarly, we can show that

$$F\left\{\frac{d^n F}{dx^n}\right\} = (is)^n F\{F(x)\} = (is)^n f(s),$$

where $F(x) = F$. Proved.

Definition: Convolution or Faltung: The convolution of two functions $F(x)$ and $G(x)$, where $-\infty < x < \infty$, is denoted and defined as

$$F * G = \int_{-\infty}^{\infty} F(u) G(x-u) du \text{ or } F * G = \int_{-\infty}^{\infty} G(u) F(x-u) du$$

Theorem (9) Convolution theorem for Fourier transform.

The Fourier transform of the convolution of $F(x)$ and $G(x)$ is the product of the Fourier transforms of $F(x)$ and $G(x)$ i.e.

$$F\{F * G\} = F\{f(x)\} \cdot F\{g(x)\}$$

Proof: $LHS = F\{F * G\} = F\left\{\int_{-\infty}^{\infty} F(u) G(x-u) du\right\}$
 $= \int_{-\infty}^{\infty} e^{-isx} \left\{ \int_{-\infty}^{\infty} F(u) G(x-u) du \right\},$ by definition of F.T.

$$= \int_{-\infty}^{\infty} F(u) \left\{ \int_{-\infty}^{\infty} e^{isx} G(x-u) du \right\} du, \text{ (by changing the order of integration)}$$

$$= \int_{-\infty}^{\infty} F(u) \left\{ \int_{-\infty}^{\infty} e^{is(u+v)} G(v) dv \right\} du, \text{ putting } x-u=v, \quad du=du$$

$$= \int_{-\infty}^{\infty} e^{isu} F(u) \left\{ \int_{-\infty}^{\infty} e^{isv} G(v) dv \right\} du$$

$$= \int_{-\infty}^{\infty} e^{isu} F(u) F\{G(v)\} du, \text{ by definition}$$

$$= \int_{-\infty}^{\infty} e^{isu} F(u) F\{G(n)\} du$$

$$= F\{G(n)\} \int_{-\infty}^{\infty} e^{isu} F(u) du$$

$$= F\{G(n)\} \cdot F\{F(u)\}, \text{ by definition}$$

$$= F\{G(n)\} \cdot F\{F(n)\} = F\{F(n)\} \cdot F\{G(n)\}.$$

Fourier Transform (S)

Proved.