

(11) Prove that

$$L^{-1} \left\{ \frac{p}{p^4 + p^2 + 1} \right\} = \frac{2}{\sqrt{3}} \sinh \frac{t}{2} \cdot \sin \frac{1}{2} \sqrt{3} t$$

Ans: We have,

$$\begin{aligned} & L^{-1} \left\{ \frac{p}{p^4 + p^2 + 1} \right\} \\ &= L^{-1} \left\{ \frac{p}{(p^2)^2 + 2p^2 + 1 - p^2} \right\} \\ &= L^{-1} \left\{ \frac{p}{(p^2 + 1)^2 - p^2} \right\} = L^{-1} \left\{ \frac{p}{\{(p^2 + 1) + p\} \{(p^2 + 1) - p\}} \right\} \\ &= L^{-1} \left\{ \frac{p}{(p^2 + p + 1)(p^2 - p + 1)} \right\} \quad \text{--- (1)} \end{aligned}$$

$$\text{Let } \frac{p}{(p^2 + p + 1)(p^2 - p + 1)} = \frac{Ap + B}{(p^2 + p + 1)} + \frac{Cp + D}{p^2 - p + 1} \quad \text{--- (2)}$$

$$\begin{aligned} \Rightarrow p &= (Ap + B)(p^2 - p + 1) + (Cp + D)(p^2 + p + 1) \\ \Rightarrow p &= (A + C)p^3 + (-A + B + C + D)p^2 \\ &\quad + (A - B + C + D)p + B + D \end{aligned}$$

Equating the like co-efficient

$$\begin{aligned} A + C &= 0 \quad \text{--- (3)} \\ -A + B + C + D &= 0 \quad \text{--- (4)} \\ A - B + C + D &= 1 \quad \text{--- (5)} \\ B + D &= 0 \quad \text{--- (6)} \end{aligned}$$

From (3) and (5)

$$-B + D = 1 \quad \text{--- (7)}$$

From (4) and (6)

$$-A + C = 0 \quad \text{--- (8)}$$

Adding (3) and (8)

$$2C = 0 \Rightarrow C = 0$$

∴ From (3)

$$A = 0$$

Adding (6) and (7)

$$2D = 1 \Rightarrow D = \frac{1}{2}$$

∴ From (7)

$$B = D - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

∴ From (1) and (2) we can write

$$\begin{aligned} L^{-1} \left\{ \frac{p}{(p^2+p+1)(p^2+p-1)} \right\} &= L^{-1} \left\{ \frac{-\frac{1}{2}}{p^2+p+1} + \frac{\frac{1}{2}}{p^2+p-1} \right\} \\ &= \frac{1}{2} L^{-1} \left\{ \frac{1}{p^2+p+1} - \frac{1}{p^2+p-1} \right\} \\ &= \frac{1}{2} L^{-1} \left\{ \frac{1}{p^2 - 2p \cdot \frac{1}{2} + (\frac{1}{2})^2 + 1 - \frac{1}{4}} \right\} \\ &\quad - \frac{1}{2} L^{-1} \left\{ \frac{1}{p^2 + 2p \cdot \frac{1}{2} + (\frac{1}{2})^2 + 1 - \frac{1}{4}} \right\} \\ &= \frac{1}{2} L^{-1} \left\{ \frac{1}{(p - \frac{1}{2})^2 + \frac{3}{4}} \right\} - \frac{1}{2} L^{-1} \left\{ \frac{1}{(p + \frac{1}{2})^2 + \frac{3}{4}} \right\} \\ &= \frac{1}{2} e^{\frac{1}{2}t} L^{-1} \left\{ \frac{1}{p^2 + \frac{3}{4}} \right\} - \frac{1}{2} e^{-\frac{1}{2}t} L^{-1} \left\{ \frac{1}{p^2 + \frac{3}{4}} \right\} \\ &= \frac{1}{2} e^{\frac{1}{2}t} \frac{\sin \frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}} - \frac{1}{2} e^{-\frac{1}{2}t} \frac{\sin \frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}} \\ &= \frac{1}{\sqrt{3}} e^{\frac{1}{2}t} \sin \frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}} e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2}t \\ &= \frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2}t \left[\frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{2} \right] \\ &= \frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2}t \cdot \sinh \frac{t}{2} \quad \text{proved} \end{aligned}$$

VWVS

Second translation or shifting theorem

Remember: If $L^{-1}\{f(p)\} = F(t)$

$$\therefore L^{-1}\{e^{-ap} f(p)\} = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$$

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(i) Find $L^{-1}\left\{\frac{(p+1)e^{-\pi p}}{p^2+p+1}\right\}$

Qm: We have

$$\begin{aligned} & L^{-1}\left\{\frac{p+1}{p^2+p+1}\right\} \\ &= L^{-1}\left\{\frac{p+1}{p^2+2\cdot p\cdot \frac{1}{2}+(\frac{1}{2})^2+1-\frac{1}{4}}\right\} \\ &= L^{-1}\left\{\frac{p+1}{(p+\frac{1}{2})^2+\frac{3}{4}}\right\} \\ &= L^{-1}\left\{\frac{(p+\frac{1}{2})+\frac{1}{2}}{(p+\frac{1}{2})^2+\frac{3}{4}}\right\} = L^{-1}\left\{\frac{p+\frac{1}{2}}{(p+\frac{1}{2})^2+\frac{3}{4}}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{(p+\frac{1}{2})^2+\frac{3}{4}}\right\} \\ &= e^{-\frac{1}{2}t} L^{-1}\left\{\frac{p}{p^2+\frac{3}{4}}\right\} + \frac{1}{2}e^{-\frac{1}{2}t} L^{-1}\left\{\frac{1}{p^2+\frac{3}{4}}\right\} \\ &= e^{-\frac{1}{2}t} \cos \frac{\sqrt{3}}{2}t + \frac{1}{2}e^{-\frac{1}{2}t} \frac{\sin \frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}} \\ &= e^{-\frac{1}{2}t} \cos \frac{\sqrt{3}}{2}t + \frac{1}{2} \times \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2}t \\ &= e^{-\frac{t}{2}} \left[\cos \frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2}t \right] \\ \therefore L^{-1}\left\{\frac{(p+1)e^{-\pi p}}{p^2+p+1}\right\} &= \begin{cases} e^{-\frac{(t-\pi)}{2}} \left[\cos \frac{\sqrt{3}}{2}(t-\pi) + \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2}(t-\pi) \right] ; \\ 0 \end{cases} \end{aligned}$$

du

Inverse Laplace transform of derivatives

Theorem: If $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\{f^n(p)\} = L^{-1}\left[\frac{d^n}{dp^n} f(p)\right] = (-1)^n t^n F(t), \quad n=1, 2, 3, \dots$$

Proof: we have,

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n} f(p) = (-1)^n f^n(p)$$

$$\therefore L^{-1}\{f^n(p)\} = L^{-1}\left[\frac{d^n}{dp^n} f(p)\right]$$

$$= (-1)^n t^n F(t)$$

$$= (-1)^n t^n L^{-1}\{f(p)\}$$

→

① Find $L^{-1}\left\{\log \frac{p+3}{p+2}\right\}$

Ans: Let $f(p) = \log \frac{p+3}{p+2}$

$$= \log(p+3) - \log(p+2)$$

$$\therefore f'(p) = \frac{1}{p+3} - \frac{1}{p+2}$$

$$\therefore L^{-1}\{f'(p)\} = L^{-1}\left\{\frac{1}{p+3}\right\} - L^{-1}\left\{\frac{1}{p+2}\right\}$$

$$\Rightarrow (-1)t L^{-1}\{f(p)\} = e^{-3t} - e^{-2t}$$

$$\Rightarrow L^{-1}\{f(p)\} = -\frac{1}{t} \{e^{-3t} - e^{-2t}\}$$

$$\Rightarrow L^{-1}\left\{\log \frac{p+3}{p+2}\right\} = \frac{1}{t} (e^{-2t} - e^{-3t})$$

Ans.