Fourier Transform (s)   Proved

Theorem (10) Parseval's identity for Fourier transform

If $f(s)$ and $g(s)$ are the complex Fourier transforms of $F(x)$ and $G(x)$ respectively, then

(i) $\frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\,\overline{g(s)}\,ds = \int_{-\infty}^{\infty} F(x)\,\overline{G(x)}\,dx$

(ii) $\frac{1}{2\pi}\int_{-\infty}^{\infty} [f(s)]^2\,ds = \int_{-\infty}^{\infty} [F(x)]^2\,dx$

where bar signifies the complex conjugate.

Proof (i) using the inversion for fourier transforms, we get, $G(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(s)\, e^{-isx}\, ds$ ——— ①

Taking complex conjugates on both sides of ①, we have

$$\overline{G(x)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g(s)}\, e^{isx}\, ds \quad \text{——— ②}$$

$$\therefore \int_{-\infty}^{\infty} F(x)\,\overline{G(x)}\,dx = \int_{-\infty}^{\infty} F(x)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g(s)}\, e^{isx}\, ds\right\}dx \quad \{\text{by } ②\}.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g(s)}\left\{\int_{-\infty}^{\infty} F(x)\, e^{isx}\, dx\right\}ds,$$

(by changing the order of integration)

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g(s)}\, f(s)\, ds \text{, by definition of F.T.}$$

Proof (ii) Taking $G(x) = F(x)$ in part (i), we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\,\overline{f(s)} = \int_{-\infty}^{\infty} F(x)\cdot\overline{F(x)}\,dx$$

$$\text{or, } \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(s)|^2 = \int_{-\infty}^{\infty} |F(x)|^2\,dx$$

Proved.

※ Some important results (Always used).

① $\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$ ② $\int_0^{\infty} e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$

③ $\Gamma(n) = a^n\int_0^{\infty} e^{-ax}\,x^{n-1}\,dx$, where $\Gamma(n) = \int_0^{\infty} e^{-x}\,x^{n-1}\,dx$.

④ $\int_0^{\infty} \frac{\sin ax}{x}\,dx = \frac{\pi}{2}$, where $a>0$

⑤ $\int_0^{\infty} e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$

(6) $\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx) + b\sin(bx)\right]$

(7) $\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx) - b\cos(bx)\right]$

Problems

Problem (1) Find the F.T. of $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$

Ans Given that $f(x) = \begin{cases} 1, & \text{for } |x| < a, -a < x < a \\ 0, & \text{for } |x| > a \end{cases}$

$$F\{f(x)\} = \int_{-\infty}^{\infty} e^{-isx} f(x)\,dx$$

$$= \int_{-\infty}^{-a} e^{-isx} f(x)\,dx + \int_{-a}^{a} e^{-isx} f(x)\,dx + \int_{a}^{\infty} e^{-isx} f(x)\,dx.$$

$$= \int_{\infty}^{a} e^{isy} f(-y)(-dy) + \int_{-a}^{a} e^{-isx}\cdot 1\cdot dx + \int_{a}^{\infty} e^{-isx}\cdot 0\cdot dx \quad \text{, where } x=-y \text{ in Ist integ.}$$

$$= \int_{a}^{\infty} e^{isy} f(-y)\,dy + \left[\frac{e^{-isx}}{-is}\right]_{x=-a}^{a} + 0$$

$$= \int_{a}^{\infty} e^{isy}\cdot 0\cdot dy + \left[\frac{e^{isa} - e^{-isa}}{is}\right] + 0$$

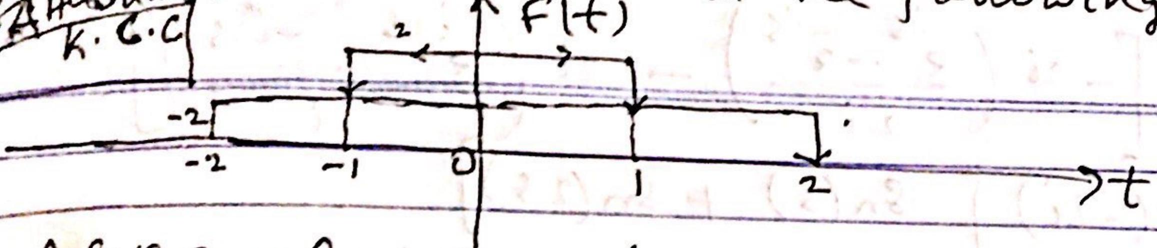
$$= 0 + \frac{2}{s}\sin sa + 0 \qquad (\because e^{i\theta} = \cos\theta + i\sin\theta)$$

$$= \frac{2}{s}\sin(sa) \quad \textbf{Ans}$$

Scanned with CamScanner

Problem (2) Find the fourier transform of the function shown in the following figure.

8A Hashmi
K.G.C



Answer from the above figure, mentioned in the question, the formation of the function is as

$$f(t) = \begin{cases} 1, & -2 \leq t \leq -1 \\ 2, & -1 \leq t \leq 1 \\ 1, & 1 \leq t \leq 2 \end{cases}$$

Now by definition of F.T., we have

$$F\{F(t)\} = \int_{-\infty}^{\infty} e^{-ist} f(t)\, dt$$

$$= \int_{-2}^{2} e^{-ist} f(t)\, dt$$

$$= \int_{-2}^{-1} e^{-ist} F(t)\,dt + \int_{-1}^{1} e^{-ist} F(t)\,dt + \int_{1}^{2} e^{-ist} F(t)\,dt$$

$$= \int_{-2}^{-1} e^{-ist} \cdot 1\, dt + \int_{-1}^{1} e^{-ist} \cdot 2\, dt + \int_{1}^{2} e^{-ist} \cdot 1\, dt$$

$$= \left[\frac{e^{-ist}}{-is}\right]_{-2}^{-1} + \left[\frac{2e^{-ist}}{-is}\right]_{-1}^{1} + \left[\frac{e^{-ist}}{-is}\right]_{1}^{2}$$

$$= \frac{i}{s}\left[e^{is} - e^{i2s}\right] + \frac{2i}{s}\left(e^{-is} - e^{is}\right) + \frac{i}{s}\left(e^{-2is} - e^{-is}\right)$$

$$= \frac{i}{s}\left[e^{is} - e^{i2s} + 2e^{-is} - 2e^{is} + e^{-2is} - e^{-is}\right]$$

$$= \frac{i}{s}\left[-e^{is} + e^{-is} - e^{i2s} + e^{-i2s}\right] = \frac{i}{s}\left[-(e^{is} - e^{-is}) - (e^{i2s} - e^{-i2s})\right]$$

$$= \frac{i}{s}\left[-2i\left(\frac{e^{is} - e^{-is}}{2i}\right) - 2i\left(\frac{e^{i2s} - e^{-i2s}}{2i}\right)\right]$$

$$= \frac{i}{s}(-2i)\left[\sin(s) + \sin(2s)\right]$$

$$= \frac{-2i^2}{s}\left(\sin s + 2\sin s \cos s\right) = \frac{2}{s}\sin s\,(1 + 2\cos s)$$

Ans

Problem (3) Show that the Fourier transform of $f(x) = e^{-x^2/2}$ is $\sqrt{2\pi}\, e^{-\frac{p^2}{2}}$.

Ans

$$F\{f(x)\} = \int_{-\infty}^{\infty} e^{-isx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{-isx} \cdot e^{-\frac{x^2}{2}}\,dx = \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2} + isx\right)}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{(x^2 + 2isx)}{2}}\,dx = \int_{-\infty}^{\infty} e^{-\frac{(x+is)^2}{2}} \cdot e^{-\frac{s^2}{2}}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{s^2}{2}} \cdot e^{-y^2} \cdot \sqrt{2}\cdot dy, \text{ where } \frac{x+is}{\sqrt{2}} = y$$

$$= \sqrt{2} \cdot e^{-\frac{s^2}{2}} \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

$$= \sqrt{2}\, e^{-\frac{s^2}{2}} \cdot 2\int_{0}^{\infty} e^{-y^2}\,dy \quad \text{(even fun)}$$

$$= 2\sqrt{2}\, e^{-\frac{s^2}{2}} \cdot \frac{\sqrt{\pi}}{2} = e^{-\frac{s^2}{2}}\sqrt{2\pi}$$

Ans