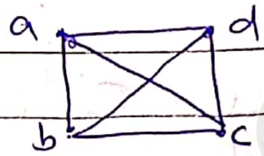
Planar Graph.

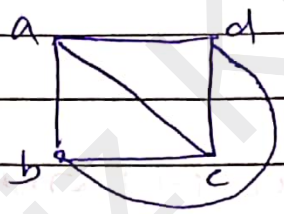
A graph G is said to be planar if it can be drawn in a plane so that no edge cross except at vertices.

For example

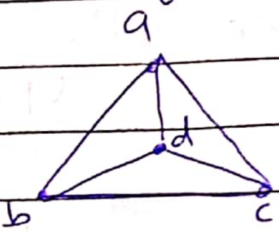
following g is complete graph of 4 vertices which can be draw as



But this can also be draw in following two ways



&



A graph which cannot be drawn without crossing of its edges is called non-planar.

A drawing of graph G on a plane surface such that no-edge intersect is called embedding of G.

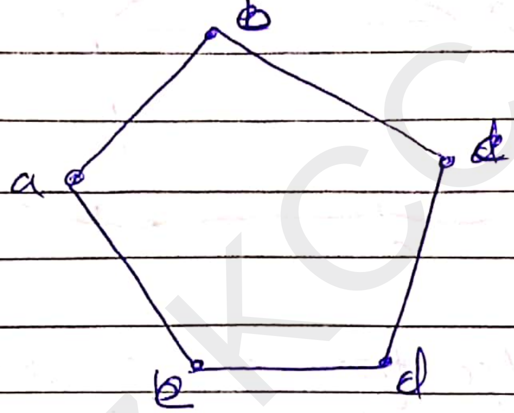
Representation of graph as G = (V, E) is called combinatorial or abstract graph

Representation of graph by figures is called geometric graph.

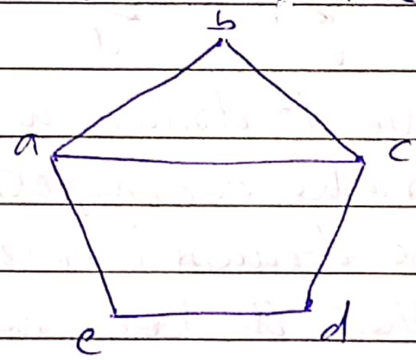
(Kuratowski's first graph)

Thm The complete graph of five vertices is non-planar.
(K_5 is non-planar)

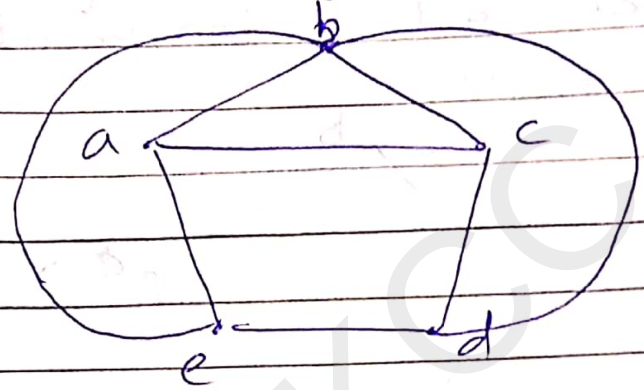
Proof Let us name five vertices of complete graph as a, b, c, d & e. Since graph is complete, there is an edge between every pair of vertices, so we must get a circuit by joining a to b, to c to d to e to a as follows



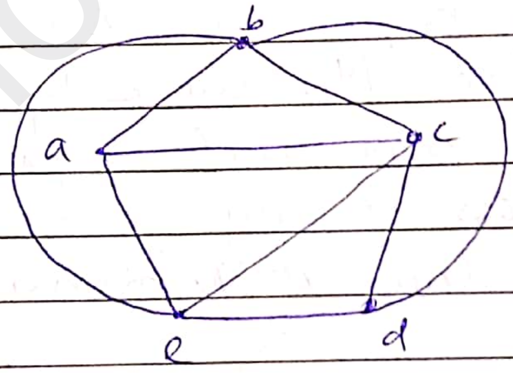
Above edges forms a pentagon which divide whole plane of paper into two regions, one inside and one outside. Now the edge from a to c can be drawn from inside or from outside (so that it does not intersect any of five edges). Suppose that we choose to draw it inside the pentagon



Now we draw two edges one from b to d & other from b to e, Since both the edges cannot be drawn inside the pentagon without crossing the edges that have already been drawn, we draw both the edges outside the pentagon.



Since edge from c to e cannot be drawn outside the pentagon without crossing any edge already draw, it is drawn inside the pentagon



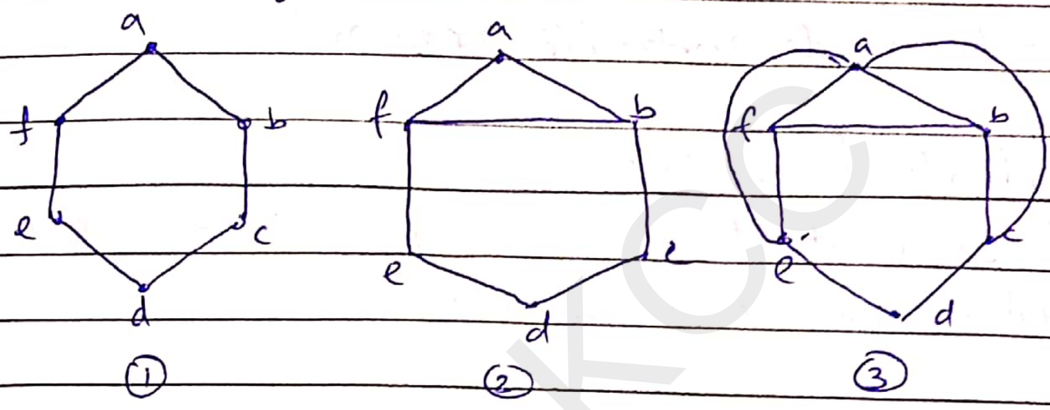
Now we have to draw an edge between a and d to make a complete graph. This edge can neither be drawn inside nor outside the pentagon so that it does not cross any edge previously drawn. So graph cannot be embedded in a plane.

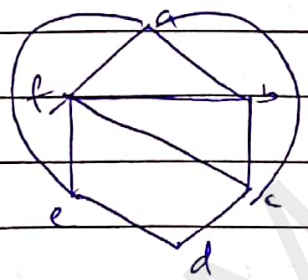
Teacher's Signature..........................

Kuratowski's second graph

Th^m A regular connected graph with six vertices and 9 edges is non-planar. ($K_{3,3}$ is nonplanar)

Proof: Same as previous problem. Just see how edges are drawn successively without crossing till no more edge can be added.





→ Now edge from b to e cannot be drawn without crossing any edge

Th^m: Any simple planar graph can be embedded in a plane such that every edge is drawn as straight line segment.

Region: A plane representation of graph divides the plane into regions (also called faces or meshes)

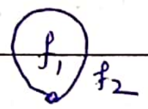
Infinite region: The portion of plane lying outside a graph embedded in a plane is called infinite region or outer region.

Thm A planar graph can be embedded in a plane in such a way that any specified region can be made infinite region.

Euler's formula:
A connected planar graph with n vertices and e edges has $e-n+2$ region.

Proof Let G be a connected planar graph. We will prove this result by induction on the number of edges of G.
Let us denote number of region by f.
For $e=1$ we have two cases

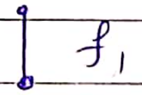
Case I
$n=1$, we have following graph



$e=1$ $n=1$

We have two regions f_1 & f_2 i.e $f=2$
and $e-n+2=2$. so statement is true.

Case II $n=2$, we have following graph



$e=1$; $n=2$

Teacher's Signature.....................

Here we have only one region ie $f=1$
Also $e-n+2 = 1-2+2 = 1$, So the statement is true for $e=1$. ~~Let us~~

Let us assume that statement is true for any planar graph with at most $e-1$ edges.

Let G be a graph with e edges and f faces or regions. If G is a tree then $e=n-1$ and $f=1$ (ie it has only one region).

So in this case $e-n+2 = (n-1)-n+2 = 1 = f$

Hence the theorem is true in this case

If G is not a tree then it has some circuit. Let a is an edge in some circuit. Removal of this edge "a" from the plane representation of graph will merge the two region into one new region. Thus the graph $G-\{a\}$ is a connected graph with n vertices $e-1$ edges and $f-1$ regions

According to our hypothesis, theorem holds for a planar graph with $e-1$ edges

ie $\quad f-1 = (e-1)-n+2$

$$\Rightarrow f = e-n+2$$

This completes the proof