Fourier Transform (s) Proved

Theorem (10) Parseval's identity for Fourier transform

If $f(s)$ and $g(s)$ are the complex Fourier transforms of $F(x)$ and $G(x)$ respectively, then

(i) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\,\overline{g(s)}\,ds = \int_{-\infty}^{\infty} F(x)\,\overline{G(x)}\,dx$$

(ii) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} [f(s)]^2\,ds = \int_{-\infty}^{\infty} [F(x)]^2\,dx.$$

where bar signifies the complex conjugate.

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Proof (i) using the inversion for fourier transforms, We get, $G(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(s)\, e^{-isx}\, ds$ ——— (1)

Taking complex conjugates on both sides of (1), we have

$$\overline{G(x)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g(s)}\, e^{isx}\, ds \quad \text{——— (2)}$$

$$\therefore \int_{-\infty}^{\infty} F(x)\overline{G(x)}\, dx = \int_{-\infty}^{\infty} F(x)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g(s)}\, e^{isx}\, ds\right\} dx \quad \{\text{by (2)}\}.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g(s)}\left\{\int_{-\infty}^{\infty} F(x)\, e^{isx}\, dx\right\} ds,$$

(by changing the order of integration)

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g(s)}\, f(s)\, ds \text{ , by definition of F.T.}$$

Proof (ii) Taking $G(x) = F(x)$ in part (i), we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\overline{f(s)} = \int_{-\infty}^{\infty} F(x)\cdot\overline{F(x)}\, dx$$

$$\text{or, } \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(s)|^2 = \int_{-\infty}^{\infty} |F(x)|^2\, dx$$

Proved.

*** Some important results (Always used).**

(1) $\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$ (2) $\int_0^{\infty} e^{-ax}\cos bx\, dx = \frac{a}{a^2+b^2}$

(3) $\Gamma(n) = a^n \int_0^{\infty} e^{-ax} x^{n-1}\, dx$, where $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\, dx$.

(4) $\int_0^{\infty} \frac{\sin ax}{x}\, dx = \frac{\pi}{2}$, where $a > 0$

(5) $\int_0^{\infty} e^{-ax}\sin bx\, dx = \frac{b}{a^2+b^2}$

⑥ $\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx) + b\sin(bx)\right]$

⑦ $\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx) - b\cos(bx)\right]$

Problems

Problem ① Find the F.T. of $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$

Ans Given that $f(x) = \begin{cases} 1, & \text{for } |x| < a,\ -a < x < a \\ 0, & \text{for } |x| > a \end{cases}$

$$F\{f(x)\} = \int_{-\infty}^{\infty} e^{-isx} f(x)\,dx$$

$$= \int_{-\infty}^{-a} e^{-isx} f(x)\,dx + \int_{-a}^{a} e^{-isx} f(x)\,dx + \int_{a}^{\infty} e^{-isx} f(x)\,dx.$$

$$= \int_{\infty}^{a} e^{isy} f(-y)(-dy) + \int_{-a}^{a} e^{-isx}\cdot 1\cdot dx + \int_{a}^{\infty} e^{-isx}\cdot 0\cdot dx$$, where $x=-y$ in Ist integ.

$$= \int_{a}^{\infty} e^{isy} f(-y)\,dy + \left[\frac{e^{-isx}}{-is}\right]_{x=-a}^{a} + 0$$

$$= \int_{a}^{\infty} e^{isy}\cdot 0\cdot dy + \left[\frac{e^{isa} - e^{-isa}}{is}\right] + 0$$

$$= 0 + \frac{2}{s}\sin sa + 0 \qquad (\because e^{i\theta} = \cos\theta + i\sin\theta)$$

$$= \frac{2}{s}\sin(sa)$$ **Ans**

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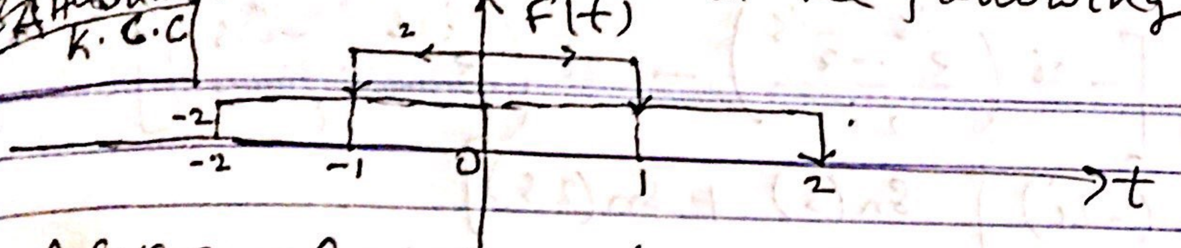
Problem ② Find the fourier transform of the function shown in the following figure.



Answer from the above figure, mentioned in the question, the formation of the function is as

$$f(t) = \begin{cases} 1 & , -2 \le t \le -1 \\ 2 & , -1 \le t \le 1 \\ 1 & , 1 \le t \le 2 \end{cases}$$

Now by definition of F.T., we have

$$F\{f(t)\} = \int_{-\infty}^{\infty} e^{-ist} f(t)\, dt$$

$$= \int_{-2}^{2} e^{-ist} f(t)\, dt$$

$$= \int_{-2}^{-1} e^{-ist} F(t)\,dt + \int_{-1}^{1} e^{-ist} F(t)\,dt + \int_{1}^{2} e^{-ist} F(t)\,dt$$

$$= \int_{-2}^{-1} e^{-ist} \cdot 1\, dt + \int_{-1}^{1} e^{-ist} \cdot 2\, dt + \int_{1}^{2} e^{-ist} \cdot 1\, dt$$

$$= \left[\frac{e^{-ist}}{-is}\right]_{-2}^{-1} + \left[\frac{2e^{-ist}}{-is}\right]_{-1}^{1} + \left[\frac{e^{-ist}}{-is}\right]_{1}^{2}$$

$$= \frac{i}{s}\left[e^{is} - e^{i2s}\right] + \frac{2i}{s}\left(e^{-is} - e^{is}\right) + \frac{i}{s}\left(e^{-2is} - e^{-is}\right)$$

$$= \frac{i}{s}\left[e^{is} - e^{i2s} + 2e^{-is} - 2e^{is} + e^{-i2s} - e^{-is}\right]$$

$$= \frac{i}{s}\left[-e^{is} + e^{-is} - e^{i2s} + e^{-i2s}\right] = \frac{i}{s}\left[-(e^{is} - e^{-is}) - (e^{i2s} - e^{-i2s})\right]$$

$$= \frac{i}{s}\left[-2i\left(\frac{e^{is} - e^{-is}}{2i}\right) - 2i\left(\frac{e^{i2s} - e^{-i2s}}{2i}\right)\right]$$

$$= \frac{i}{s}(-2i)\left[\sin(s) + \sin(2s)\right]$$

$$= \frac{-2i^2}{s}\left(\sin s + 2\sin s \cos s\right) = \frac{2}{s}\sin s\,(1 + 2\cos s)$$

Ans

Problem (3) Show that the Fourier transform of $f(x) = e^{-x^2/2}$ is $\sqrt{2\pi}\, e^{-\frac{p^2}{2}}$.

Ans

$$F\{f(x)\} = \int_{-\infty}^{\infty} e^{-isx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{-isx} \cdot e^{-\frac{x^2}{2}} dx = \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2} + isx\right)} dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{(x^2 + 2isx)}{2}} dx = \int_{-\infty}^{\infty} e^{-\frac{(x+is)^2}{2}} \cdot e^{-\frac{s^2}{2}} dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{s^2}{2}} \cdot e^{-y^2} \cdot \sqrt{2} \cdot dy, \text{ where } \frac{x+is}{\sqrt{2}} = y$$

$$= \sqrt{2} \cdot e^{-\frac{s^2}{2}} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \sqrt{2}\, e^{-\frac{s^2}{2}} \cdot 2\int_{0}^{\infty} e^{-y^2} dy \quad \text{(even fun)}$$

$$= 2\sqrt{2}\, e^{-\frac{s^2}{2}} \cdot \frac{\sqrt{\pi}}{2} = e^{-\frac{s^2}{2}} \sqrt{2\pi}$$

Ans