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FOURIER TRANSFORM (6)

Lem (4): Find the fourier transform of $f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

Given that $f(x) = \begin{cases} 1-x^2, & -1 \le x \le 1 \\ 0, & |x| > 1 \end{cases}$

$$F\{f(x)\} = \int_{-\infty}^{\infty} e^{-isx} f(x)\, dx$$

$$= \int_{-\infty}^{-1} e^{-isx} f(x)\, dx + \int_{-1}^{1} e^{-isx} f(x)\, dx + \int_{1}^{\infty} e^{-isx} f(x)\, dx$$

put $x = -y$

$$= \int_{\infty}^{1} e^{isy} f(-y)(-dy) + \int_{-1}^{1} e^{-isx}(1-x^2)\, dx + \int_{1}^{\infty} e^{-isx} \cdot 0\, dx$$

$$= \int_{1}^{\infty} e^{isy} f(-y)\, dy + \int_{-1}^{1} e^{-isx}(1-x^2)\, dx + 0$$

$$= 0 + \int_{-1}^{1} e^{-isx}(1-x^2)\, dx = \int_{-1}^{1} e^{-isx}(1-x^2)\, dx$$

$$= \left[(1-x^2)\frac{e^{-isx}}{-is}\right]_{-1}^{1} - \int_{-1}^{1}(-2x)\left[\frac{e^{-isx}}{-is}\right] dx$$

$$= 0 - \frac{2}{is}\int_{-1}^{1} x e^{-isx}\, dx$$

$$= -\frac{2}{is}\left[x\frac{e^{-isx}}{-is} - \int 1 \cdot \frac{e^{-isx}}{-is}\, dx\right]_{-1}^{1}$$

$$= -\frac{2}{is}\left[\frac{x e^{-isx}}{-is} + \frac{1}{is}\frac{e^{-isx}}{-is}\right]_{-1}^{1}$$

$$= -\frac{2}{is}\left[\frac{e^{-is}}{-is} + \frac{e^{+is}}{-is} + \frac{1}{is}\left(\frac{e^{-is} - e^{is}}{-is}\right)\right]$$

$$= \frac{2i^2}{is}\left[\frac{e^{is}+e^{-is}}{-is} + \frac{e^{-is}-e^{is}}{s^2}\right]$$

$$= \frac{2i}{s}\left[\frac{e^{is}+e^{-is}}{2}\times\frac{2}{-is} - \frac{e^{is}-e^{-is}}{2i}\times\frac{2i}{s^2}\right]$$

$$= \frac{2i}{s}\left[\cos s \times \frac{-2}{is} - \sin s \times \frac{2i}{s^2}\right]$$

$$= \frac{2i}{s}\left[\cos s \times \frac{2i}{s} - \frac{2i}{s^2}\sin s\right]$$

$$= \frac{4i^2}{s^3}\left[s^2\cos s - \sin s\right]$$

$$= \frac{-4}{s^3}\left(s^2\cos s - \sin s\right) = \frac{4}{s^3}\left(-s^2\cos s + \sin s\right)$$

Problem (5) Find fourier sine transform of $e^{-|x|}$. Ans

Hence evaluate $\int_0^\infty \frac{x\sin(mx)}{1+x^2}\,dx$

Ans: $\because |x| = x$ in $(0, \infty)$.

Now $F_s\{F(x)\} = f_s(s) = \int_0^\infty F(x)\sin(sx)\,dx$

$$= \int_0^\infty e^{-|x|}\sin(sx)\,dx = \int_0^\infty e^{-x}\sin(sx)\,dx$$

$$= \frac{s}{1+s^2} \text{ Ans } \left(\because \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}\right).$$

Now for second part:

$$F(x) = F_s^{-1}\{f_s(s)\} = \frac{2}{\pi}\int_0^\infty f_s(s)\cdot\sin(sx)\,ds$$

$$\Rightarrow F(x) = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2}\sin(sx)\,ds$$

$$\Rightarrow \frac{\pi}{2} f(x) = \int_0^\infty \frac{s}{1+s^2} \sin(sx)\, ds$$

$$\Rightarrow \frac{\pi}{2} e^{-x} = \int_0^\infty \frac{s}{1+s^2} \sin(sx)\, ds$$

Replacing x by m. we have

$$\frac{\pi}{2} e^{-m} = \int_0^\infty \frac{s}{1+s^2} \sin(ms)\, ds$$

$$\Rightarrow \int_0^\infty \left(\frac{x}{1+x^2}\right) \sin(mx)\, dx = \frac{\pi}{2} e^{-m}$$

Ans

Problem ⑥ Find the fourier transform of $f(x) = e^{-a|x|}$, where $a > 0$ and $x \in (-\infty, \infty)$.

Ans: $\because |x| = \begin{cases} -x & \text{if } x \le 0 \\ x & \text{if } x > 0 \end{cases}$

$$F\{f(x)\} = \bar{f}(s) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx = \int_{-\infty}^{\infty} e^{-a|x|}\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{0} e^{-a|x|}\, e^{-isx}\, dx + \int_0^\infty e^{-a|x|}\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{0} e^{ax}\, e^{-isx}\, dx + \int_0^\infty e^{-ax}\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{0} e^{(a-is)x}\, dx + \int_0^\infty e^{-(a+is)x}\, dx$$

$$= \left[\frac{e^{(a-is)x}}{a-is}\right]_{x=-\infty}^{0} + \left[\frac{e^{-(a+is)x}}{-(a+is)}\right]_{x=0}^{x=\infty}$$

$$= \left[\frac{1}{a-is} - \frac{0}{a-is}\right] - \left[\frac{0}{a+is} - \frac{1}{a+is}\right]$$

$$= \frac{1}{a-is} + \frac{1}{a+is} = \frac{2a}{a^2+s^2} \quad \text{Ans}$$

Problem (7): Find the F.T. of $f(x) = \begin{cases} x, & |x| \leq a \\ 0, & |x| > a \end{cases}$

Ans: Given that $f(x) = \begin{cases} x, & -a \leq x \leq a \\ 0, & |x| > a \end{cases}$

Now $F\{f(t)\} = \int_{-\infty}^{\infty} e^{-ist} f(t)\, dt.$

$$= \int_{-\infty}^{-a} e^{-ist} f(t)\, dt + \int_{-a}^{a} e^{-ist} f(t)\, dt + \int_{a}^{\infty} e^{-ist} f(t)\, dt$$

put $t = -y$

$$= \int_{\infty}^{a} e^{isy} f(-y)\, d(-y) + \int_{-a}^{a} e^{-ist} f(t)\, dt + \int_{a}^{\infty} e^{-ist} f(t)\, dt$$

$$= \int_{a}^{\infty} e^{isy} . 0\, dy + \int_{-a}^{a} e^{-ist} . t\, dt + \int_{a}^{\infty} e^{-ist} . 0\, dt$$

$$= 0 + \left(\frac{e^{-ist}}{-is} . t\right)_{-a}^{a} - \int_{-a}^{a} \frac{e^{-ist}}{-is} . 1 . dt$$

$$= \left(\frac{a}{-is}\right)\left(e^{-isa} + e^{isa}\right) + \frac{1}{is}\left(\frac{e^{-ist}}{-is}\right)_{-a}^{a}$$

$$= \frac{ia}{s} 2\cos(sa) + \frac{1}{s^2}\left(e^{-isa} - e^{isa}\right)$$

$$= \frac{2ias}{s^2} \cos(sa) - \frac{2i}{s^2}\left(\frac{e^{isa} - e^{-isa}}{2i}\right)$$

$$= \frac{2i}{s^2}\left[as . \cos(sa) - \sin(sa)\right] \quad \text{Ans.}$$