$\Rightarrow L^{-1}\{f(p)\} = -\frac{2}{t}(\cos t - 1)$

$\Rightarrow L^{-1}\{\log(1+\frac{1}{p^2})\} = \frac{2}{t}(1-\cos t)$

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Convolution (or Falting) (Defn):

Let $F(t)$ and $G(t)$ be two functions of class A, then the convolution of the two functions $F(t)$ and $G(t)$ denoted by $F*G$ is defined by the relation

$$F*G = \int_0^t F(x)\,G(t-x)\,dx$$

This relation $F*G$ is also called the resultant or Falting of F and G.

(i) $F*G$ is commutative i.e; $F*G = G*F$

Proof: $F*G = \int_0^t F(x)\,G(t-x)\,dx$

Putting $t-x = y$

$-dx = dy$

when $x = 0$ then $y = t$

" $x = t$ " $y = 0$

$$\therefore F*G = \int_t^0 -F(t-y)\,G(y)\,dy$$

$$= \int_0^t F(t-y)\,G(y)\,dy$$

$$= G*F$$

(ii) $F*G$ is associative i.e; $(F*G)*H = F*(G*H)$

(iii) F^*G is distributive w.r.to addition

ie; $F^*(G+H) = F^*G + F^*H$

Proof:

$$F^*(G+H) = \int_0^t F(x)\left[G(t-x) + H(t-x)\right] dx$$

$$= \int_0^t F(x)\, G(t-x)\, dx + \int_0^t F(x)\, H(t-x$$

$$= F^*G + F^*H$$

—x—

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Convolution theorem (convolution property):
[or Falting theorem]

Statement: Let $F(t)$ and $G(t)$ be two functions class A and let $L^{-1}\{f(p)\} = F(t)$ and $L^{-1}\{g(p)\} = G(t)$ then

$$L^{-1}\{f(p)\, g(p)\} = \int_0^t F(x)\, G(t-x)\, dx = F^*G$$

Proof:

Let $\phi(t) = \int_0^t F(x)\, G(t-x)\, dx$

We have to prove

$$L\left[\int_0^t F(x)\, G(t-x)\, dx\right] = f(p)\, g(p)$$

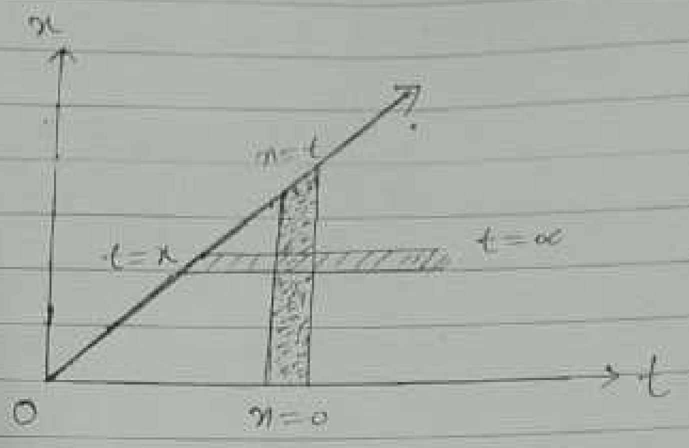
ie; $L\{\phi(t)\} = f(p)\cdot g(p)$

Now,

$$L\{\phi(t)\} = \int_0^\infty e^{-pt}\, \phi(t)\, dt$$

$$\Rightarrow L\{\phi(t)\} = \int_0^\infty e^{-pt}\left[\int_0^t F(x)\, G(t-x)\, dx\right] d$$

$$\Rightarrow L\{\phi(t)\} = \int_{t=0}^{t=\infty}\int_{x=0}^{x=t} e^{-pt} F(x)\, G(t-x)\, dx\, dt$$



NOTE:
$x=0$ ie, t-axis and $x=t$ is the line through origin

The domain of integration for this double integral is the entire area lying between the lines $x=0$ and $x=t$.

On changing the order of integration we get

$$L\{\phi(t)\} = \int_{x=0}^{\infty}\int_{t=x}^{\infty} e^{-pt} F(x)\, G(t-x)\, dx\, dt$$

$$= \int_0^{\infty}\int_x^{\infty} e^{-px} \cdot e^{-pt+px} F(x)\, G(t-x)\, dx\, dt$$

$$= \int_0^{\infty} e^{-px} F(x) \left[\int_x^{\infty} e^{-p(t-x)} G(t-x)\, dt\right] dx$$

putting $t-x = y$

$\therefore dt = dy$

when $t = x$ then $y = 0$

" $t = \infty$ " $y = \infty$

$$\therefore L\{\phi(t)\} = \int_0^{\infty} e^{-px} F(x) \left[\int_0^{\infty} e^{-py} G(y)\, dy\right] dx$$

$$\therefore L\{\phi(t)\} = \int_0^\infty e^{-px} F(x)\,dx \cdot \int_0^\infty e^{-py} G(y)\,dy$$
$$= L\{F(x)\} \cdot L\{G(y)\}$$

$$\therefore L\{\phi(t)\} = f(p) \cdot g(p)$$
$$\Rightarrow L^{-1}\{f(p) \cdot g(p)\} = \phi(t)$$
$$\Rightarrow L^{-1}\{f(p)\, g(p)\} = \int_0^t F(x)\, G(t-x)\,dx = F * G$$

proved

Problem based on convolution theorem

① Solve : $L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\}$

Ans:

Since $L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\}$

$$= L^{-1}\left\{\frac{p}{p^2+a^2} \cdot \frac{1}{p^2+a^2}\right\} = L^{-1}\{f(p) \cdot g(p)\}$$

where $f(p) = \frac{p}{p^2+a^2}$, $g(p) = \frac{1}{p^2+a^2}$

$$\therefore L^{-1}\{f(p)\} = L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at = F(t)$$

and $L^{-1}\{g(p)\} = L^{-1}\left\{\frac{1}{p^2+a^2}\right\} = \frac{1}{a}\sin at = G(t)$

Hence by convolution theorem, we have

$$L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\} = \int_0^t \cos ax \cdot \frac{1}{a}\sin a(t-x)\,dx$$

$$L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\} = \frac{1}{a}\int_0^t \cos ax\,(\sin at\cdot\cos ax - \cos at\cdot\sin ax)\,dx$$

$$= \frac{1}{a}\int_0^t \sin at\cos^2 ax\,dx - \frac{1}{a}\int_0^t \cos at\sin ax\cos ax\,dx$$

$$= \frac{\sin at}{a}\int_0^t \cos^2 ax\,dx - \frac{\cos at}{a}\int_0^t \sin ax\cos ax\,dx$$

$$= \frac{\sin at}{2a}\int_0^t (1+\cos 2ax)\,dx - \frac{\cos at}{2a}\int_0^t \sin 2ax\,dx$$

$$= \frac{\sin at}{2a}\left[x + \frac{\sin 2ax}{2a}\right]_0^t - \frac{\cos at}{2a}\left[-\frac{\cos 2ax}{+2a}\right]_0^t$$

$$= \frac{\sin at}{2a}\left[t + \frac{\sin 2at}{2a} - 0\right] - \frac{\cos at}{2a}\left[-\frac{\cos 2at}{2a} + \frac{1}{2a}\right]$$

$$= \frac{t\sin at}{2a} + \frac{\sin at\cdot\sin 2at}{4a^2} + \frac{\cos at\cdot\cos 2at}{4a^2} - \frac{\cos at}{4a^2}$$

$$= \frac{t\sin at}{2a} + \frac{\sin at\; 2\sin at\cdot\cos at}{4a^2} - \frac{\cos at}{4a^2}(1-\cos 2at)$$

$$= \frac{t\sin at}{2a} + \frac{\sin at\; 2\sin at\cos at}{4a^2} - \frac{\cos at\; 2\sin^2 at}{4a^2}$$

$$= \frac{t\sin at}{2a}$$

—x—

Solve : $L^{-1}\left\{\frac{1}{(p-1)^5(p+2)}\right\}$

$$L^{-1}\left\{\frac{1}{(p-1)^5(p+2)}\right\} = L^{-1}\left\{\frac{1}{(p-1)^5(p-1+3)}\right\}$$

$$\therefore L^{-1}\left\{\frac{1}{p^5(p+3)}\right\} = \frac{t^4}{72} - \frac{t^3}{54} + \frac{e^{-3t}}{18}\left(t^2 \frac{e^{3t}}{3} - 0\right) - \frac{e^{-3t}}{18} \times \frac{2}{3}\int_0^t x e^{3x}\,dx$$

$$= \frac{t^4}{72} - \frac{t^3}{54} + \frac{t^2}{54} - \frac{e^{-3t}}{27}\left[x\frac{e^{3x}}{3}\right]_0^t + \frac{e^{-3t}}{27}\int_0^t \frac{e^{3x}}{3}\,dx$$

$$= \frac{t^4}{72} - \frac{t^3}{54} + \frac{t^2}{54} - \frac{e^{-3t}}{27}\left[t\frac{e^{3t}}{3} - 0\right] + \frac{e^{-3t}}{27} \times \frac{1}{3}\left[\frac{e^{3x}}{3}\right]_0^t$$

$$= \frac{t^4}{72} - \frac{t^3}{54} + \frac{t^2}{54} - \frac{t}{81} + \frac{e^{-3t}}{81}\left[\frac{e^{3t}}{3} - \frac{1}{3}\right]$$

$$= \frac{t^4}{72} - \frac{t^3}{54} + \frac{t^2}{54} - \frac{t}{81} + \frac{1}{243} - \frac{e^{-3t}}{243}$$

$\therefore$ (1) becomes

$$\therefore L^{-1}\left\{\frac{1}{(p-1)^5(p+2)}\right\} = e^{t}\left[\frac{t^4}{72} - \frac{t^3}{54} + \frac{t^2}{54} - \frac{t}{81} + \frac{1}{243} - \frac{e^{-3t}}{243}\right]$$

Ans

IMP

(3) Find $L^{-1}\left\{\frac{1}{(p+1)(p^2+1)}\right\}$

Ans:-

$$L^{-1}\left\{\frac{1}{(p+1)(p^2+1)}\right\}$$

$$= L^{-1}\left\{\frac{1}{(p+1)} \cdot \frac{1}{(p^2+1)}\right\} = L^{-1}\{f(p) \cdot g(p)\}$$

where $f(p) = \frac{1}{p+1}$, $g(p) = \frac{1}{p^2+1}$

$$\therefore L^{-1}\{f(p)\} = L^{-1}\left\{\frac{1}{p+1}\right\} = e^{-t} = F(t)$$

and $L^{-1}\{g(p)\} = L^{-1}\left\{\frac{1}{p^2+1}\right\} = \sin t = G(t)$.