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FOURIER TRANSFORM (7)

Problem (8) find the inverse F.T. of $\bar{f}(p) = e^{-|p|\gamma}$,
where $\gamma \in (-\infty, \infty)$.

Ans: $|p| = \begin{cases} -p, & p \leq 0 \\ p, & p \geq 0 \end{cases}$

$$F^{-1}\{\bar{f}(p)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(p) e^{ipx} dp$$

$$= \frac{1}{2\pi} \left[\int_{-\infty}^0 \bar{f}(p) e^{ipx} dp + \int_0^{\infty} \bar{f}(p) e^{ipx} dp \right]$$

$$= \frac{1}{2\pi} \left[\int_{-\infty}^0 e^{-|p|\gamma} \cdot e^{ipx} dp + \int_0^{\infty} e^{-|p|\gamma} \cdot e^{ipx} dp \right]$$

$$= \frac{1}{2\pi} \left[\int_{-\infty}^0 e^{p\gamma} \cdot e^{ipx} dp + \int_0^{\infty} e^{-p\gamma} \cdot e^{ipx} dp \right]$$

$$= \frac{1}{2\pi} \left[\int_{-\infty}^0 e^{(y+ix)p} dp + \int_0^{\infty} e^{-(y-ix)p} dp \right]$$

$$= \frac{1}{2\pi} \left[\left\{ \frac{e^{(y+ix)p}}{y+ix} \right\}_{-\infty}^0 + \left\{ \frac{e^{-(y-ix)p}}{-(y-ix)} \right\}_0^{\infty} \right]$$

$$= \frac{1}{2\pi} \left[\frac{1}{y+ix} (1-0) + \frac{1}{ix-y} (0-1) \right]$$

$$= \frac{1}{2\pi} \left[\frac{1}{ix+y} - \frac{1}{ix-y} \right]$$

$$= \frac{1}{2\pi} \left[\frac{ix-y-ix-y}{-x^2-y^2} \right] = \frac{\gamma}{\pi(x^2+y^2)} \quad \underline{\underline{\text{Ans}}}$$

Problem (9) Prove that the Fourier transform of the step function $f(x) = \begin{cases} \frac{\sqrt{2\pi}}{2l}, & -l < x < l \\ 0, & |x| > l \end{cases}$ is $\frac{\sqrt{2\pi}}{2l} \sin sl$.

Answer $F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-isx} dx$

$$= \int_{-\infty}^{-l} f(x) e^{-isx} dx + \int_{-l}^l f(x) e^{-isx} dx + \int_l^{\infty} f(x) e^{-isx} dx$$

Put $x = -y$

$$= \int_{+\infty}^{+l} f(-y) e^{isy} (-dy) + \int_{-l}^l \frac{\sqrt{2\pi}}{2l} e^{-isx} dx + \int_l^{\infty} 0 \cdot e^{-isx} dx$$

$$= \int_l^{\infty} 0 \cdot e^{isy} dy + \frac{\sqrt{2\pi}}{2l} \left[\frac{e^{-isx}}{-is} \right]_{x=-l}^{x=l} + 0$$

$$= 0 + \frac{\sqrt{2\pi}}{2l} \left[\frac{e^{-isl} - e^{isl}}{-is} \right] + 0$$

$$= \frac{\sqrt{2\pi}}{2l} \left[\frac{e^{isl} - e^{-isl}}{2 \cdot i} \times \frac{2i}{is} \right]$$

$$= \frac{\sqrt{2\pi}}{2l} \cdot \sin sl \times \frac{2}{s} = \frac{\sqrt{2\pi}}{sl} \cdot \sin sl$$

Problem (10) Solve the integral equation Ans

$$\int_0^{\infty} f(x) \cos sx \, dx = e^{-s}$$

Ans: $\therefore \int_0^{\infty} f(x) \cos sx \, dx = e^{-s}$ (1)

By definition. $F_c\{f(x)\} = \int_0^{\infty} f(x) \cos sx \, dx = \bar{f}_c(s)$ (2)

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$$\text{and } F_c^{-1}\{\bar{f}_c(s)\} = f(x) = \frac{2}{\pi} \int_0^{\infty} \bar{f}_c(s) \cos sx \, ds \quad (3)$$

By (1) & (2), we have

$$\bar{f}_c(s) = \bar{e}^s.$$

Now by (3), we have

$$\begin{aligned} f(x) &= \frac{2}{\pi} \int_0^{\infty} \bar{e}^s \cos sx \, ds \\ &= \frac{2}{\pi} \cdot \frac{1}{1+x^2} \left(\because \int_0^{\infty} \bar{e}^{ax} \cos bx \, dx = \frac{a}{a^2+b^2} \right) \end{aligned}$$

$$\therefore f(x) = \frac{2}{\pi(1+x^2)} \quad \underline{\text{Ans}}$$

Problem (11) Solve the integral equation

$$\int_0^{\infty} F(\lambda) \cos \lambda x \, d\lambda = \begin{cases} 1-\lambda, & \text{for } 0 \leq \lambda < 1 \\ 0, & \text{for } \lambda > 1 \end{cases}$$

Answer By definition

$$F_c\{F(\lambda)\} = \int_0^{\infty} F(\lambda) \cos \lambda x \, d\lambda = \bar{f}_c(\lambda) \quad (7)$$

$$\text{Then } \bar{f}_c(\lambda) = \begin{cases} 1-\lambda, & \text{for } 0 \leq \lambda \leq 1 \\ 0, & \text{for } \lambda > 1 \end{cases}$$

The inverse formula relative to (1) is

$$F_c^{-1}\{\bar{f}_c(\lambda)\} = F(x) = \frac{2}{\pi} \int_0^{\infty} \bar{f}_c(\lambda) \cos \lambda x \, d\lambda.$$

From which, we have

$$\frac{\pi}{2} F(x) = \int_0^{\infty} \bar{f}_c(\lambda) \cos \lambda x \, d\lambda.$$

$$\Rightarrow \frac{\hat{A}}{2} F(x) = \int_0^1 f_c(\lambda) \cos \lambda x d\lambda + \int_1^\infty f_c(\lambda) \cos \lambda x d\lambda$$

$$= \int_0^1 (1-\lambda) \cos \lambda x d\lambda + \int_1^\infty 0 \cdot \cos \lambda x d\lambda$$

$$= \int_0^1 (1-\lambda) \cos \lambda x d\lambda + 0$$

$$= \left[(1-\lambda) \frac{\sin \lambda x}{x} - \int (-1) \frac{\sin \lambda x}{x} d\lambda \right]_0^1$$

$$= \left[(1-\lambda) \frac{\sin \lambda x}{x} + \frac{1}{x} \cdot \left(-\frac{\cos \lambda x}{x} \right) \right]_0^1$$

$$= 0 - \frac{1}{x^2} (\cos x - 1)$$

$$\frac{\hat{A}}{2} F(x) = \frac{1 - \cos x}{x^2}$$

$$\Rightarrow f(x) = \frac{2}{\hat{A} x^2} (1 - \cos x) \text{ Ans}$$

Problem (12) Find the Fourier sine transform of $f(x) = \frac{1}{x}$.

Ans Let $F_s\{f(s)\} = \bar{f}_s(s)$. Then

$$\bar{f}_s(s) = \int_0^\infty f(x) \sin sx dx = \int_0^\infty \frac{\sin sx}{x} dx = \frac{\hat{A}}{2} \quad \text{--- (1)}$$

$$\therefore \bar{f}_s(s) = \frac{\hat{A}}{2} \left[\text{Since } \int_0^\infty \frac{\sin mx}{x} dx = \frac{\hat{A}}{2}, \text{ if } m > 0 \right]$$

Ans