Hence by convolution theorem we have

$$L^{-1}\left\{\frac{1}{(p+1)}\cdot\frac{1}{(p^2+1)}\right\} = \int_0^t e^{-x}\sin(t-x)\,dx = I \text{ (say)} \quad (1)$$

$$= \left[\sin(t-x)\frac{e^{-x}}{-1}\right]_0^t - \int_0^t -\cos(t-x)\frac{e^{-x}}{-1}dx$$

[Int. by parts]

$$= 0 + \sin t - \int_0^t e^{-x}\cos(t-x)\,dx$$

$$= \sin t - \left[\cos(t-x)\frac{e^{-x}}{-1}\right]_0^t + \int_0^t \sin(t-x)\frac{e^{-x}}{-1}dx$$

$$= \sin t + \left[e^{-t} - \cos t\right] - \int_0^t e^{-x}\sin(t-x)dx$$

$$\Rightarrow \quad I = \sin t + e^{-t} - \cos t - I \quad [\text{From } (1)]$$

$$\Rightarrow \quad 2I = e^{-t} + \sin t - \cos t$$

$$\Rightarrow \quad I = \frac{1}{2}\left[e^{-t} + \sin t - \cos t\right]$$

$$\Rightarrow L^{-1}\left\{\frac{1}{(p+1)(p^2+1)}\right\} = \frac{1}{2}\left[e^{-t} + \sin t - \cos t\right] \text{ Ans.}$$

(4) Use the convolution theorem to find

(1) $L^{-1}\left\{\frac{1}{(p+1)(p-2)}\right\}$

Ans: $L^{-1}\left\{\frac{1}{(p+1)(p-2)}\right\}$

$$= L^{-1}\left\{\frac{1}{(p+1)}\cdot\frac{1}{(p-2)}\right\} = L^{-1}\{f(p)\cdot g(p)\}$$

where $f(p) = \frac{1}{p+1}$, and $g(p) = \frac{1}{p-2}$

$$\therefore L^{-1}\{f(p)\} = L^{-1}\left\{\frac{1}{p+1}\right\} = e^{-t} = F(t)$$

and $L^{-1}\{g(p)\} = L^{-1}\left\{\frac{1}{p-2}\right\} = e^{2t} = G(t)$

$\therefore$ By convolution theorem, we have

$$L^{-1}\left\{\frac{1}{(p+1)}\cdot\frac{1}{(p-2)}\right\} = \int_0^t e^{-x}\, e^{2(t-x)}\, dx$$

$$= \int_0^t e^{-x}\cdot e^{2t}\cdot e^{-2x}\, dx$$

$$= e^{2t}\int_0^t e^{-3x}\, dx$$

$$= e^{2t}\left[\frac{e^{-3x}}{-3}\right]_0^t = \frac{e^{2t}}{-3}\left[e^{-3t}-1\right]$$

$$= -\frac{e^{-t}}{3} + \frac{e^{2t}}{3} = \frac{1}{3}\left[e^{2t} - e^{-t}\right]$$ Ans.

—x—

5) Solve: $L^{-1}\left\{\frac{p+1}{(p^2+2p+2)^2}\right\}$

Ans:

$$L^{-1}\left\{\frac{p+1}{(p^2+2p+2)^2}\right\}$$

$$= L^{-1}\left\{\frac{p+1}{(p^2+2p\cdot 1+1+1)^2}\right\} = L^{-1}\left\{\frac{p+1}{\{(p+1)^2+1\}^2}\right\}$$

$$= e^{-t}\, L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} \quad \text{--- (1)}$$

Now,

$$L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = L^{-1}\left\{\frac{p}{p^2+1}\cdot\frac{1}{p^2+1}\right\}$$

$$= L^{-1}\{f(p)\cdot g(p)\}$$ where $f(p) = \frac{p}{p^2 \ldots}$

$g(p) = \frac{1}{p^2+1}$

$\therefore L^{-1}\{f(p)\} = L^{-1}\left\{\frac{p}{p^2+1}\right\} = \cos t = F(t)$

and $L^{-1}\{g(p)\} = L^{-1}\left\{\frac{1}{p^2+1}\right\} = \sin t = G(t)$

Hence by convolution theorem, we have

$$L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \int_0^t \cos x \cdot \sin(t-x)\,dx$$

$$= \frac{1}{2}\int_0^t 2\cos x \sin(t-x)\,dx$$

$$= \frac{1}{2}\int_0^t \{\sin(x+t-x) - \sin(x-t+x)\}\,dx$$

$$= \frac{1}{2}\int_0^t \{\sin t - \sin(2x-t)\}\,dx$$

$$= \frac{1}{2}\sin t \int_0^t dx - \frac{1}{2}\int_0^t \sin(2x-t)\,dx$$

$$= \frac{1}{2}\sin t \cdot [x]_0^t - \frac{1}{2}\left[-\frac{\cos(2x-t)}{2}\right]_0^t$$

$$= \frac{t}{2}\sin t + \frac{1}{4}[\cos t - \cos t]$$

$$= \frac{1}{2}t\sin t$$

$\therefore$ (1) becomes

$$L^{-1}\left\{\frac{p+1}{(p^2+2p+2)^2}\right\} = e^{-t}\cdot\frac{1}{2}t\sin t$$

$$= \frac{1}{2}te^{-t}\sin t \quad \text{Ans}$$

—x—

Solve: $L^{-1}\left\{\frac{p}{(p^2+4)^2}\right\}$

Do same as question no. (1)

(7) Solve: $L^{-1}\left\{\frac{1}{(p+a)^3}\right\}$

Ans. $L^{-1}\left\{\frac{1}{(p+a)^3}\right\}$

$= L^{-1}\left\{\frac{1}{(p+a)^2}\cdot\frac{1}{(p+a)}\right\}$

$= L^{-1}\left\{f(p)\cdot g(p)\right\}$ where $f(p)=\frac{1}{(p+a)^2}$

and $g(p)=\frac{1}{(p+a)}$

$\therefore L^{-1}\{f(p)\} = L^{-1}\left\{\frac{1}{(p+a)^2}\right\} = e^{-at}L^{-1}\left\{\frac{1}{p^2}\right\} = e^{-at}t$

and $L^{-1}\{g(p)\} = L^{-1}\left\{\frac{1}{p+a}\right\} = e^{-at}$

Hence by convolution theorem we have,

$L^{-1}\left\{\frac{1}{(p+a)^2}\cdot\frac{1}{(p+a)}\right\} = \int_0^t e^{-ax}\cdot x\cdot e^{-a(t-x)}\,dx$

$= \int_0^t x e^{-ax-at+ax}\,dx$

$= e^{-at}\int_0^t x\,dx = e^{-at}\left[\frac{x^2}{2}\right]_0^t$

$= e^{-at}\left(\frac{t^2}{2}-0\right)$

$= \frac{t^2 e^{-at}}{2}$. Ans

—x—

(8) Solve: $L^{-1}\left\{\frac{1}{(p+a)^n}\right\}$

Ans:- Do same as (7). Here only take

$f(p)=\frac{1}{(p+a)^{n-1}}$ and $g(p)=\frac{1}{(p+a)}$.

(or) $(n-2)$ and 2 (or) $(n-3)$ and 3. and so on

✓✓✓

(9) Prove that

$$L^{-1}\left\{\frac{4p+5}{(p-1)^2(p+2)}\right\} = 3te^{t} + \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t}$$

Ans:

$$L^{-1}\left\{\frac{4p+5}{(p-1)^2(p+2)}\right\}$$

$$= L^{-1}\left\{\frac{4p+5}{(p-1)^2}\cdot\frac{1}{p+2}\right\} = L^{-1}\left\{f(p)\cdot g(p)\right\}$$

where $f(p) = \frac{4p+5}{(p-1)^2}$

$g(p) = \frac{1}{p+2}$

$$\therefore L^{-1}\{f(p)\} = L^{-1}\left\{\frac{4p+5}{(p-1)^2}\right\} = L^{-1}\left\{\frac{4p}{(p-1)^2}\right\} + 5L^{-1}\left\{\frac{1}{(p-1)^2}\right\}$$

$$= 4L^{-1}\left\{\frac{(p-1)+1}{(p-1)^2}\right\} + 5L^{-1}\left\{\frac{1}{(p-1)^2}\right\}$$

$$= 4L^{-1}\left\{\frac{1}{(p-1)}\right\} + 4L^{-1}\left\{\frac{1}{(p-1)^2}\right\} + 5L^{-1}\left\{\frac{1}{(p-1)^2}\right\}$$

$$= 4e^{t} + 9e^{t}L^{-1}\left\{\frac{1}{p^2}\right\}$$

$$= 4e^{t} + 9e^{t}\cdot t$$

and

$$L^{-1}\{g(p)\} = L^{-1}\left\{\frac{1}{p+2}\right\} = e^{-2t}$$

Hence by Convolution theorem, we have

$$L^{-1}\left\{\frac{4p+5}{(p-1)^2(p+2)}\right\} = \int_0^t (4e^{x} + 9e^{x}x)e^{-2(t-x)}dx$$

$$= \int_0^t \left(4e^{x-2t+2x} + 9xe^{x-2t+2x}\right)dx$$

$$= 4e^{-2t}\int_0^t e^{3x}dx + 9e^{-2t}\int_0^t xe^{3x}dx$$

$$\therefore L^{-1}\left\{\frac{4p+5}{(p-1)^2(p+2)}\right\} = 4e^{-2t}\left[\frac{e^{3x}}{3}\right]_0^t + 9e^{-2t}\left[x\frac{e^{3x}}{3}\right]_0^t - 9e^{-2t}\int_0^t \frac{e^{3x}}{3}dx$$

$$= \frac{4}{3}e^{-2t}\left[e^{3t}-1\right] + 3e^{-2t}\left[te^{3t}-0\right] - 3e^{-2t}\left[\frac{e^{3x}}{3}\right]_0^t$$

$$= \frac{4}{3}e^{t} - \frac{4}{3}e^{-2t} + 3e^{t}t - \frac{3}{3}e^{-2t}\left[e^{3t}-1\right]$$

$$= \frac{4}{3}e^{t} - \frac{4}{3}e^{-2t} + 3te^{t} - e^{t} + e^{-2t}$$

$$= 3te^{t} + \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t}$$ proved

—x—

NOTE:

$$L\{F^n(t)\} = p^n L\{F(t)\} - p^{n-1}F(0) - p^{n-2}F'(0) - \cdots - F^{n-1}(0)$$

(i). $L\{F'''(t)\} = p^3 L\{F(t)\} - p^2F(0) - pF'(0) - F''(0)$

(ii) $L\{y'''(x)\} = p^3 L\{y(x)\} - p^2y(0) - py'(0) - y''(0)$