FOURIER TRANSFORM (8)

Problem (13) Find sine and cosine transform of $x^n e^{-ax}$

Ans Here $f(x) = x^n e^{-ax}$

$$F_c\{f(x)\} = f_c(s) = \int_0^\infty x^n e^{-ax} \cos(sx)\,dx \quad \text{—(1)}$$

and $$F_s\{f(x)\} = f_s(s) = \int_0^\infty x^n e^{-ax} \sin(sx)\,dx \quad \text{—(2)}$$

We know that $\int_0^\infty e^{-kx} x^{n-1}\,dx = \frac{\Gamma(n)}{k^n}$

$$\therefore \int_0^\infty e^{-isx}(x^n e^{-ax})\,dx = \int_0^\infty e^{-x(a+is)} x^{(n+1)-1}\,dx$$

$$= \frac{\Gamma(n+1)}{(a+is)^{n+1}}$$

Taking $a+is = re^{i\theta}$, then we get

$$\int_0^\infty e^{-isx}(x^n e^{-ax})\,dx = \frac{\Gamma(n+1)}{(re^{i\theta})^{n+1}} = \frac{\lfloor n}{r^{n+1}} e^{-i(n+1)\theta}$$

$$\Rightarrow \int_0^\infty (x^n e^{-ax})(\cos sx - i\sin sx)\,dx = \frac{\lfloor n}{(a^2+s^2)^{n+1/2}}\{\cos(n+1)\theta - i\sin(n+1)\theta\}$$

Equating real and imaginary parts.

$$\text{R.P} \Rightarrow \int_0^\infty (x^n e^{-ax})\cos(sx)\,dx = \frac{\lfloor n \cos(n+1)\theta}{(a^2+s^2)^{n+1/2}}$$

$$\text{I.P.} \Rightarrow \int_0^\infty (x^n e^{-ax})\sin(sx)\,dx = \frac{\lfloor n \sin(n+1)\theta}{(a^2+s^2)^{\frac{n+1}{2}}}$$

$$\therefore\ f_c(s) = \frac{\lfloor n \cos(n+1)\theta}{(a^2+s^2)^{\frac{n+1}{2}}}$$

$$\text{and}\quad f_s(s) = \frac{\lfloor n \sin(n+1)\theta}{(a^2+s^2)^{\frac{n+1}{2}}}$$

Ans

Problem (14) Find Fourier Sine and Cosine transform of $2e^{-5x}+5e^{-2x}$.

Ans (i) $F_c\{2e^{-5x}+5e^{-2x}\} = 2F_c\{e^{-5x}\}+5F_c\{e^{-2x}\}$

$$= 2\int_0^\infty e^{-5x}\cos sx\,dx + 5\int_0^\infty e^{-2x}\cos sx\,dx$$

$$= 2\cdot\frac{5}{5^2+s^2} + 5\cdot\frac{2}{2^2+s^2}\quad\left(\because \int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}\right)$$

$$= 10\left(\frac{1}{s^2+25}+\frac{1}{s^2+4}\right)\ \text{Ans}$$

(ii) $F_s\{2e^{-5x}+5e^{-2x}\} = 2F_s\{e^{-5x}\}+5F_s\{e^{-2x}\}$

$$= 2\int_0^\infty e^{-5x}\sin sx\,dx + 5\int_0^\infty e^{-2x}\sin sx\,dx$$

$$= 2\cdot\frac{s}{s^2+5^2} + 5\cdot\frac{s}{s^2+2^2}\quad\left(\because \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}\right)$$

$$= s\left(\frac{2}{s^2+25}+\frac{5}{s^2+4}\right)$$

$\therefore$ Sine transform is $s\left(\frac{2}{s^2+25}+\frac{5}{s^2+4}\right)$

and Cosine transform is $10\left(\frac{1}{s^2+25}+\frac{1}{s^2+4}\right)$ Ans

Question (15) find $f(x)$ if its Fourier sine transform is $\frac{s}{1+s^2}$

If $\bar{f}_s(s) = \frac{s}{1+s^2} = F_s\{f(x)\}$, then find $f(x)$.

OR, find $F_s^{-1}\left\{\frac{s}{1+s^2}\right\}$.

Answer

$$\because f(x) = F_s^{-1}\{\bar{f}_s(s)\} = F_s^{-1}\left\{\frac{s}{1+s^2}\right\}$$

$$\therefore f(x) = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2}\cdot \sin sx .\, ds$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^\infty \frac{s^2}{s(1+s^2)}\cdot \sin sx\, ds = \frac{2}{\pi}\int_0^\infty \frac{(s^2+1-1)}{s(1+s^2)}\cdot \sin sx\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{(s^2+1)-1}{s(s^2+1)}\cdot \sin sx\, ds.$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s}\, ds - \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(s^2+1)}\, ds.$$

$$= \frac{2}{\pi}\cdot\frac{\pi}{2} - \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(1+s^2)}\, ds. \quad \text{—— (1)}$$

$$\Rightarrow \frac{df(x)}{dx} = 0 - \frac{2}{\pi}\int_0^\infty \frac{s.\cos sx}{s(1+s^2)}\, ds$$

$$\Rightarrow \frac{df}{dx} = -\frac{2}{\pi}\int_0^\infty \frac{\cos sx}{1+s^2}\, ds \quad \text{—— (2)}$$

$$\Rightarrow \frac{d^2f}{dx^2} = -\frac{2}{\pi}\int_0^\infty -s.\frac{\sin sx}{1+s^2}\, ds = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2}\sin sx\, ds$$

$$= f$$

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$$\Rightarrow \frac{d^2f}{dx^2} - f = 0 \Rightarrow (D^2-1)f = 0$$

A.E. is $m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

$\therefore$ Complementry solution is

$$f = Ae^{x} + Be^{-x} \quad \text{——— (3)}$$

$$\Rightarrow \frac{df}{dx} = Ae^{x} - Be^{-x} \quad \text{——— (4)}$$

~~when $x=0$ then $f=1$ [by (1)].~~

From these conditions, (3) is

$$A + B = 1 \quad \text{——— (5)}$$

Now by (2), when $x = 0$

$$\frac{df}{dx} = -\frac{2}{\pi}\left[\tan^{-1}s\right]_{s=0}^{\infty} = -1$$

$$\therefore \frac{df}{dx} = -1, \text{ when } x = 0$$

Now by (4), applying these conditions

$$A - B = -1 \quad \text{——— (6)}$$

Solving (5) & (6), we have

$A = 0$, $B = 1$.

Now by (3), we have

$$f = e^{-x} \quad \underline{\text{Ans}}$$