Application of Laplace Transform to solution of differential equation

(1) Solve $\frac{d^2y}{dt^2} + y = 0$ under the conditions that $y = 1$, $\frac{dy}{dt} = 0$ when $t = 0$.

Ans:- We have,

$$\frac{d^2y}{dt^2} + y = 0$$

$$\Rightarrow y'' + y = 0$$

Taking laplace transform of both sides

$$\therefore L\{y''\} + L\{y\} = L\{0\}$$

$$\Rightarrow p^2 L\{y\} - py(0) - y'(0) + L\{y\} = 0$$

$$\Rightarrow (p^2+1) L\{y\} - p \times 1 - 0 = 0$$

$$\Rightarrow (p^2+1) L\{y\} = p$$

$$\Rightarrow L\{y\} = \frac{p}{p^2+1}$$

$$\Rightarrow y = L^{-1}\left\{\frac{p}{p^2+1}\right\} = \cos t$$

$\therefore$ $y = \cos t$ is required solution

—x—

(2). Solve: $(D^2+m^2)x = a\cos nt$; $t>0$, x and ... to x_0 and x_1 when $t=0$, $n \neq m$.

Ans: We have

$$(D^2+m^2)x = a\cos nt$$

$$\Rightarrow D^2x + m^2x = a\cos nt$$

$$\Rightarrow x'' + m^2x = a\cos nt$$

where x

Taking Laplace transform on both side

$$L\{x''\} + m^2 L\{x\} = a L\{\cos nt\}$$

$$\Rightarrow p^2 L\{x\} - p x(0) - x'(0) + m^2 L\{x\} = a \frac{p}{p^2+n^2}$$

$$\Rightarrow \{p^2 + m^2\} L\{x\} - p \times x_0 - x_1 = \frac{ap}{p^2+n^2}$$

$$\Rightarrow (p^2+m^2) L\{x\} = \frac{ap}{p^2+n^2} + p x_0 + x_1$$

$$\Rightarrow L\{x\} = \frac{ap}{(p^2+n^2)(p^2+m^2)} + \frac{p}{p^2+m^2} x_0 + \frac{1}{p^2+m^2} x_1$$

$$\Rightarrow x = a L^{-1}\left\{\frac{p}{(p^2+n^2)(p^2+m^2)}\right\} + x_0 L^{-1}\left\{\frac{p}{p^2+m^2}\right\} + x_1 L^{-1}\left\{\frac{1}{p^2+m^2}\right\}$$

$$\Rightarrow x = a L^{-1}\left\{\frac{p}{(p^2+n^2)(p^2-m^2)}\right\} + x_0 \cos mt + x_1 \frac{\sin mt}{m} \quad (1)$$

For $L^{-1}\left\{\frac{p}{(p^2+n^2)(p^2+m^2)}\right\}$

$$L^{-1}\left\{\frac{p}{(p^2+n^2)(p^2+m^2)}\right\} = L^{-1}\left\{\frac{p}{p^2+n^2} \cdot \frac{1}{p^2+m^2}\right\}$$

$$= L^{-1}\{f(p) \cdot g(p)\}$$

where $f(p) = \frac{p}{p^2+n^2}$, $g(p) = \frac{1}{p^2+m^2}$

$$\therefore L^{-1}\{f(p)\} = L^{-1}\left\{\frac{p}{p^2+n^2}\right\} = \cos nt$$

and $L^{-1}\{\theta(p)\} = L^{-1}\left\{\frac{1}{p^2+m^2}\right\} = \frac{1}{m}\sin mt$

Hence by convolution theorem we have

$$L^{-1}\left\{\frac{p}{(p^2+n^2)}\cdot\frac{1}{(p^2+m^2)}\right\} = \int_0^t \cos nx \cdot \frac{1}{m}\sin m(t-x)\,dx$$

$$= \frac{1}{m}\int_0^t \cos nx \cdot \sin(mt-mx)\,dx$$

$$= \frac{1}{2m}\int_0^t 2\cos nx \cdot \sin(mt-mx)\,dx$$

$$= \frac{1}{2m}\int_0^t \{\sin(nx+mt-mx) - \sin(nx-mt+mx)\}\,dx$$

$$= \frac{1}{2m}\int_0^t \left[\sin\{mt-x(m-n)\} - \sin\{x(m+n)-mt\}\right]dx$$

$$= \frac{1}{2m}\left[\frac{-\cos\{mt-x(m-n)\}}{-(m-n)}\right]_0^t - \frac{1}{2m}\left[\frac{-\cos\{x(m+n)-mt\}}{(m+n)}\right]_0^t$$

$$= \frac{1}{2m(m-n)}\left[\cos(\cancel{mt}-\cancel{mt}+nt) - \cos mt\right] + \frac{1}{2m(m+n)}\left[\cos(\cancel{mt}+nt-\cancel{mt}) - \cos mt\right]$$

$$= \frac{1}{2m(m-n)}\left[\cos nt - \cos mt\right] + \frac{1}{2m(m+n)}\left[\cos nt - \cos mt\right]$$

$$= \frac{(\cos nt - \cos mt)}{2m}\left[\frac{1}{(m-n)} + \frac{1}{(m+n)}\right]$$

$$\therefore L^{-1}\left\{\frac{p}{p^2+n^2}\cdot\frac{1}{p^2+m^2}\right\} = \frac{\cos nt - \cos mt}{2m}\left[\frac{\cancel{m}+\cancel{n}+m-\cancel{n}}{m^2-n^2}\right]$$

$$= \frac{\cos nt - \cos mt}{\cancel{2}\cancel{m}}\cdot\frac{\cancel{2m}}{m^2-n^2}$$

$$= \frac{1}{(m^2-n^2)}\cos nt - \frac{1}{(m^2-n^2)}\cos mt$$

$\therefore$ (1) becomes

$$\therefore x = a\left[\frac{1}{(m^2-n^2)}\cos nt - \frac{1}{(m^2-n^2)}\cos mt\right] + x_0\cos mt + x_1\frac{\sin mt}{m}$$

$$= x_0\cos mt + x_1\frac{1}{m}\sin mt + \frac{a}{(m^2-n^2)}\cos nt - \frac{a}{(m^2-n^2)}\cos mt$$

Ans.

—x—

(3) Solve: $(D+2)^2 y = 4e^{-2t}$, $y(0) = -1$ and $y'(0) = 4$

Ans:

We have,

$$(D+2)^2 y = 4e^{-2t}$$

$$\Rightarrow (D^2+4D+4)y = 4e^{-2t}$$

$$\Rightarrow D^2y + 4Dy + 4y = 4e^{-2t}$$

$$\Rightarrow y'' + 4y' + 4y = 4e^{-2t}$$ where $y'' = D^2y = \frac{d^2y}{dt^2}$

Taking Laplace transform of both side, we get

$\therefore L\{y''\} + 4L\{y'\} + 4L\{y\} = 4L\{e^{2t}\}$

$\Rightarrow p^2L\{y\} - p\,y(0) - y'(0) + 4[pL\{y\} - y(0)] + 4L\{y\}$
$= 4 \cdot \frac{1}{p+2}$

$\Rightarrow (p^2+4p+4)L\{y\} - p\,y(0) - y'(0) - 4\,y(0) = \frac{4}{p+2}$

$\Rightarrow (p^2+4p+4)L\{y\} - p\times(-1) - 4 - 4\times(-1) = \frac{4}{p+2}$

$\Rightarrow (p^2+4p+4)L\{y\} + p - \cancel{4} + \cancel{4} = \frac{4}{p+2}$

$\Rightarrow (p^2+4p+4)L\{y\} = \frac{4}{p+2} - p = \frac{4-p^2-2p}{p+2}$

$\Rightarrow L\{y\} = \frac{4-p^2-2p}{(p+2)} \cdot \frac{1}{(p^2+4p^2+4)}$

$= \frac{4-p^2-2p}{(p+2)^3} = \frac{-\{p^2+2p-4\}}{(p+2)^3}$

$= \frac{-\{p^2+2\cdot p\cdot 2+4-2p-8\}}{(p+2)^3}$

$= \frac{-\{(p+2)^2\}+2p+8}{(p+2)^3}$

$= \frac{-(p+2)^2+2(p+2)+4}{(p+2)^3}$

$\Rightarrow y = L^{-1}\left[\frac{-(p+2)^2+2(p+2)+4}{(p+2)^3}\right]$

$\Rightarrow y = e^{-2t}L^{-1}\left[\frac{-p^2+2p+4}{(p)^3}\right]$ [By shifting theorem]

$\therefore y = e^{-2t} L^{-1}\left\{\frac{4}{p^3} - \frac{1}{p} + \frac{2}{p^2}\right\}$

$= e^{-2t}\left\{4 \cdot \frac{t^2}{\lfloor 2} - 1 + 2\frac{t}{\lfloor 1}\right\}$

$= e^{-2t}\left\{\frac{4t^2}{2} - 1 + 2t\right\}$

$= e^{-2t}\left\{2t^2 + 2t - 1\right\}$ Ans

—x—

(4) Solve $(D^2-2D+2)y = 0$, $y = Dy = 1$ when $t=0$

Ans:

We have,

$(D^2-2D+2)y = 0$

$\Rightarrow D^2y - 2Dy + 2y = 0$

$\Rightarrow y'' - 2y' + 2y = 0$ where $D \equiv \frac{d}{dt}$

Taking Laplace transform of both sides, we get

$\therefore L\{y''\} - 2L\{y'\} + 2L\{y\} = L\{0\}$

$\Rightarrow p^2 L\{y\} - p\,y(0) - y'(0) - 2pL\{y\} - 2\,y(0)$
$+ 2L\{y\} = 0$

$\Rightarrow (p^2-2p+2)L\{y\} - p\,y(0) - y'(0) + 2\,y(0) = 0$

$\Rightarrow (p^2-2p+2)L\{y\} = p + 1 - 2$

$\Rightarrow L\{y\} = \frac{p-1}{p^2-2p+2}$

$\Rightarrow L\{y\} = \frac{p-1}{p^2-2\cdot p\cdot 1+1^2+1}$

$\Rightarrow y = L^{-1}\left\{\frac{p-1}{(p-1)^2+1}\right\}$

$\Rightarrow y = e^{t} L^{-1}\left\{\frac{p}{p^2+1}\right\}$ {By shifting theorem

$\Rightarrow y = e^{t}\cos t$

Which is the required solution.

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(5) Solve: $(D^2+9)y = \cos 2t$ if $y(0)=1$, $y(\frac{\pi}{2})=-1$

We have,

$(D^2+9)y = \cos 2t$

$\Rightarrow D^2y + 9y = \cos 2t$

$\Rightarrow y'' + 9y = \cos 2t$

Taking laplace transform on both side, we get

$\therefore L\{y''\} + 9L\{y\} = L\{\cos 2t\}$

$\Rightarrow p^2L\{y\} - py(0) - y'(0) + 9L\{y\} = \frac{p}{p^2+2^2}$

$\Rightarrow (p^2+9)L\{y\} - p\times 1 - A = \frac{p}{p^2+4}$ {Let $y'(0)=A$

$\Rightarrow (p^2+9)L\{y\} - p - A = \frac{p}{p^2+4}$

$\Rightarrow (p^2+9)L\{y\} = \frac{p}{p^2+4} + p + A$

$\Rightarrow L\{y\} = \frac{p}{(p^2+4)(p^2+9)} + \frac{p}{p^2+9} + \frac{A}{p^2+9}$

$\Rightarrow y = L^{-1}\left\{\frac{p}{(p^2+4)(p^2+9)}\right\} + L^{-1}\left\{\frac{p}{p^2+9}\right\} + AL^{-1}\left\{\frac{1}{p^2\right.$

$\Rightarrow y = L^{-1}\left\{\frac{p}{(p^2+4)(p^2+9)}\right\} + \cos 3t + A\frac{1}{3}\sin 3t$

For $L^{-1}\left\{\frac{p}{(p^2+4)(p^2+9)}\right\}$

$$L^{-1}\left\{\frac{p}{(p^2+4)(p^2+9)}\right\} = L^{-1}\left\{\frac{p}{p^2+4}\cdot\frac{1}{p^2+9}\right\}$$

$$= L^{-1}\left\{f(p)\cdot g(p)\right\}$$

where $f(p)=\frac{p}{p^2+4}$, $g(p)=\frac{1}{p^2+9}$

$$\therefore L^{-1}\left\{f(p)\right\} = L^{-1}\left\{\frac{p}{p^2+4}\right\} = \cos 2t$$

and $L^{-1}\left\{g(p)\right\} = L^{-1}\left\{\frac{1}{p^2+9}\right\} = \frac{1}{3}\sin 3t$

Hence by convolution theorem we have,

$$L^{-1}\left\{\frac{p}{(p^2+4)}\cdot\frac{1}{p^2+9}\right\} = \int_0^t \cos 2x \cdot \frac{1}{3}\sin 3(t-x)\,dx$$

$$= \frac{1}{6}\int_0^t 2\cos 2x \cdot \sin(3t-3x)\,dx$$

$$= \frac{1}{6}\int_0^t \left\{\sin(2x+3t-3x) - \sin(2x-3t+3x)\right\} dx$$

$$= \frac{1}{6}\int_0^t \left\{\sin(3t-x) - \sin(5x-3t)\right\} dx$$

$$= \frac{1}{6}\left[\frac{\cos(3t-x)}{1} + \frac{\cos(5x-3t)}{5}\right]_0^t$$

$$= \frac{1}{6}\left[\cos 2t + \frac{\cos 2t}{5} - \cos 3t - \frac{\cos 3t}{5}\right]$$

$\therefore L^{-1}\left\{\frac{p}{(p^2+4)(p^2+9)}\right\} = \frac{1}{6}\left[\frac{6\cos 2t}{5} - \frac{6\cos 3t}{5}\right]$

$= \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t$

$\therefore$ (1) becomes

$y = \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t + \cos 3t + \frac{A}{3}\sin 3t$

$= \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{A}{3}\sin 3t$ ——— (2)

Since $y(\frac{\pi}{2}) = -1$

Putting $t = \frac{\pi}{2}$ in (2)

$\therefore \quad -1 = \frac{1}{5}\cos(2\frac{\pi}{2}) + \frac{4}{5}\cos(3\frac{\pi}{2}) + \frac{A}{3}\sin(3\frac{\pi}{2})$

$\Rightarrow -1 = -\frac{1}{5} + \frac{4}{5}\times 0 + \frac{A}{3}\times(-1)$

$\Rightarrow \frac{A}{3} = -\frac{1}{5} + 1 = \frac{4}{5}$

$\Rightarrow A = \frac{12}{5}$

$\therefore$ (2) becomes

$y = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{12}{15}\sin 3t$

which is the required solution.

—x—

Solve : $(D+1)^2 y = t$ given that $y=-3$ when $t=0$, $y=-1$ when $t=1$

We have,