Partial Differential Equations with three or more independent variables (Jacobi's method)

Let $f(x_1, x_2, x_3, p_1, p_2, p_3) = 0$ —— ① where $p_i = \frac{\partial z}{\partial x_i}$ $(i=1,2,3)$

be the differential eqn where the dependent variable z does not appear in any term except in $p_i\left(\frac{\partial z}{\partial x_i}\right)$.

Then the subsidiary equations are

$$\frac{dx_1}{-\frac{\partial f}{\partial p_1}} = \frac{dp_1}{\frac{\partial f}{\partial x_1}} = \frac{dx_2}{-\frac{\partial f}{\partial p_2}} = \frac{dp_2}{\frac{\partial f}{\partial x_2}} = \frac{dx_3}{-\frac{\partial f}{\partial p_3}} = \frac{dp_3}{\frac{\partial f}{\partial x_3}} \quad \text{—— ②}$$

Now, find two independent integrals $F_1 = a_1$ & $F_2 = a_2$ of eqns ②. If these satisfy the condition

$$(F_1, F_2) = \sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\cdot\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\cdot\frac{\partial F_2}{\partial x_r}\right) = 0 \quad \text{—— ③}$$

and if p_1, p_2, p_3 can be found as functions of x_1, x_2, x_3 from $f=0$, $F_1 = a_1$ & $F_2 = a_2$ then substituting these values in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$ and on integrating we get the complete integral.

Example: Apply Jacobi's method to find complete integral of $p_1^3 + p_2^2 + p_3 - 1 = 0$

Solⁿ Here $f \equiv p_1^3 + p_2^2 + p_3 - 1 = 0$

The subsidiary equations are

$$\frac{dx_1}{-3p_1^2} = \frac{dp_1}{0} = \frac{dx_2}{-2p_2} = \frac{dp_2}{0} = \frac{dx_3}{-1} = \frac{dp_3}{0}$$

We can take two equations $dp_1 = 0$ & $dp_2 = 0$

Which on solving, we get $p_1 = a_1$ & $p_2 = a_2$

Hence we have $F_1 \equiv p_1 = a_1$ & $F_2 \equiv p_2 = a_2$

Clearly
$$(F_1 F_2) = \left(\frac{\partial F_1}{\partial x_1}\cdot\frac{\partial F_2}{\partial p_1} - \frac{\partial F_1}{\partial p_1}\cdot\frac{\partial F_2}{\partial x_1}\right) + \left(\frac{\partial F_1}{\partial x_2}\cdot\frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2}\cdot\frac{\partial F_2}{\partial x_2}\right) + \left(\frac{\partial F_1}{\partial x_3}\cdot\frac{\partial F_2}{\partial p_3} - \frac{\partial F_1}{\partial p_3}\cdot\frac{\partial F_2}{\partial x_3}\right)$$

$$= 0.$$

Now we solve equations
$$p_1 = a_1$$
$$p_2 = a_2$$
$$\& \quad p_1^3 + p_2^2 + p_3 - 1 = 0$$

We get $\quad p_1 = a_1, \; p_2 = a_2$ & $p_3 = 1 - (a_1^3 + a_2^2)$

Putting these values in
$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$$

We get
$$dz = a_1 dx_1 + a_2 dx_2 + (1 - a_1^3 - a_2^2) dx_3$$

On integrating we get
$$z = a_1 x_1 + a_2 x_2 + (1 - a_1^3 - a_2^2) x_3 + a_3$$

Which is the complete integral.

Ans

Ex Find the complete integral of

$$p_3x_3(p_1+p_2)+x_1+x_2=0.$$

Sol^n Here $f \equiv p_3x_3(p_1+p_2)+x_1+x_2=0$ ——— ①

The subsidiary equations are

$$\frac{dx_1}{-p_3x_3}=\frac{dp_1}{1}=\frac{dx_2}{-p_3x_3}=\frac{dp_2}{1}=\frac{dx_3}{-x_3(p_1+p_2)}=\frac{dp_3}{p_3(p_1+p_2)}$$

Taking 2nd & 4th members, we have

$$dp_1=dp_2$$

$$\Rightarrow p_1-p_2=a_1$$

ie $F_1 \equiv p_1-p_2=a_1$ ——— ②

Taking 5th & 6th members, we have

$$\frac{dx_3}{-x_3}=\frac{dp_3}{p_3}$$

ie $\frac{dp_3}{p_3}+\frac{dx_3}{x_3}=0$

$$\Rightarrow \log p_3+\log x_3=\log a_2$$

$$\Rightarrow \log p_3x_3=\log a_2$$

$$\Rightarrow p_3x_3=a_2$$

ie $F_2 \equiv p_3x_3=a_2$ ——— ③

Here also $(F_1,F_2)=0$.

Now solving ①, ② and ③

From $p_3x_3(p_1+p_2)=-x_1-x_2$

$$\Rightarrow a_2(p_1+p_2)=-x_1-x_2$$

$$\Rightarrow p_1+p_2=-\left(\frac{x_1+x_2}{a_2}\right)$$ ——— ④

Now, from ② & ④

$$p_1 + p_2 = -\left(\frac{x_1+x_2}{a_2}\right)$$

$$p_1 - p_2 = a_1$$

$$2p_1 = a_1 - \left(\frac{x_1+x_2}{a_2}\right)$$

$$\Rightarrow p_1 = \frac{a_1}{2} - \frac{x_1+x_2}{2a_2} \quad \text{——— (5)}$$

Putting in ② $p_1 - p_2 = a_1$

$$\Rightarrow \frac{a_1}{2} - \frac{x_1+x_2}{2a_2} - p_2 = a_1$$

$$\Rightarrow p_2 = -\frac{a_1}{2} - \frac{x_1+x_2}{2a_2} \quad \text{——— (6)}$$

and from ③ $p_3 = \frac{a_2}{x_3}$ ——— (7)

Substituting values of p_1, p_2 & p_3 in

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$$

$$\Rightarrow dz = \left(\frac{a_1}{2} - \frac{x_1+x_2}{2a_2}\right) dx_1 + \left(-\frac{a_1}{2} - \frac{x_1+x_2}{2a_2}\right) dx_2 + \frac{a_2}{x_3} dx_3$$

$$\Rightarrow dz = \frac{a_1}{2}\left(dx_1 - dx_2\right) - \left(\frac{x_1+x_2}{2a_2}\right)\left(dx_1 + dx_2\right) + \frac{a_2}{x_3} dx_3$$

$$\Rightarrow z = \frac{a_1}{2}\left(x_1 - x_2\right) - \frac{1}{4a_2}\left(x_1+x_2\right)^2 + a_2 \log x_3 + a_3$$

$$\Rightarrow 4a_2 z = 2a_1 a_2 (x_1 - x_2) - (x_1+x_2)^2 + 4a_2^2 \log x_3 + 4a_2 a_3$$

Which is the required complete integral.